# SCHWARZ-PICK-TYPE ESTIMATES FOR THE HYPERBOLIC DERIVATIVE

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We obtain Schwarz-Pick-type estimates for the hyperbolic derivative of an analytic self-map of the unit disk in $\mathbb{C}$.



## 1. Preliminaries

We denote by $\Delta$ the open unit disk in $\mathbb{C}$, and for $z \in \Delta$, we denote by $\phi_z \in \operatorname{Aut}(\Delta)$ the automorphism which interchanges 0 and $z$: $\phi_z(\lambda) = (z-\lambda)/(1-\bar{z}\lambda)$. We denote by $\rho$ the hyperbolic distance on $\Delta$:

$$\rho(\lambda,z) = \tanh^{-1}\left|\phi_z(\lambda)\right| = \frac{1}{2}\log\frac{1+\left|\phi_z(\lambda)\right|}{1-\left|\phi_z(\lambda)\right|}. \tag{1.1}$$

The following is a well-known consequence of the maximum principle.

Schwarz's Lemma 1.1. *Let $f:\Delta\to\Delta$ be analytic with $f(0)=0$. Then*

$$\left|f(\lambda)\right| \le |\lambda|, \quad \text{that is, } \rho\big(f(\lambda),f(0)\big) \le \rho(\lambda,0)\ \forall\lambda\in\Delta. \tag{1.2}$$

Consequently, we have also $|f'(0)| \le 1$. To remove the normalization $f(0)=0$, one may consider the function

$$g = \phi_{f(z)} \circ f \circ \phi_z, \tag{1.3}$$

which has

$$g(0) = 0, \qquad g'(0) = \frac{f'(z)\big(1-|z|^2\big)}{1-\left|f(z)\right|^2} \tag{1.4}$$

to obtain the following.

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Schwarz-Pick Lemma 1.2. *Let $f : \Delta \to \Delta$ be analytic. Then,*

$$|\phi_{f(z)} \circ f(\lambda)| \leq |\phi_z(\lambda)|, \quad \text{that is, } \rho(f(\lambda), f(z)) \leq \rho(\lambda, z) \ \forall \lambda, z \in \Delta. \tag{1.5}$$

Consequently, $f^*(z) := g'(0)$ has $|f^*(z)| \leq 1$, and so $\rho(f^*(z), \cdot)$ is defined on $\Delta$, as long as $f$ is not an automorphism—for in this case, $|f^*| \equiv 1$. As such, we are interested in the following two results.

Theorem 1.3 (see [6]). *Let $f : \Delta \to \Delta$ be analytic, and not an automorphism. Then*

$$|\rho(0, f^*(\lambda)) - \rho(0, f^*(z))| \leq 2\rho(\lambda, z) \quad \forall \lambda, z \in \Delta. \tag{1.6}$$

*So, for example, if $f^*(\lambda)$ and $f^*(z)$ are on the same side of a ray emanating from the origin, then $\rho(f^*(\lambda), f^*(z)) \leq 2\rho(\lambda, z)$.*

Theorem 1.4 (see [1]). *Let $f : \Delta \to \Delta$ be analytic, not an automorphism, with $f(0) = 0$. Then*

$$\rho(f^*(0), f^*(z)) \leq 2\rho(0, z) \quad \forall z \in \Delta. \tag{1.7}$$

In the next section of this paper, we employ a procedure which yields simple proofs of Theorems 1.3 and 1.4 and extends these results. In particular, Theorem 1.4 is not applicable if $f(0) \neq 0$, as the function $\exp((\lambda + 1)/(\lambda - 1))$ shows. Below however, we obtain a version (Proposition 2.3) which removes the normalization and applies at any pair of points in $\Delta$, thus furnishing a more complete analog of Schwarz-Pick Lemma 1.2 for $f^*$. In the final section, we obtain some further related results.

We will use the following easily verified facts.

(A) Schwarz-Pick Lemma 1.2 and a little manipulation reveal that $f(\lambda)$ lies in the closed disk with center $c = f(z)(1 - |\phi_z(\lambda)|^2)/(1 - |f(z)|^2|\phi_z(\lambda)|^2)$ and radius $r = |\phi_z(\lambda)|(1 - |f(z)|^2)/(1 - |f(z)|^2|\phi_z(\lambda)|^2)$. Consequently, $|c| - r \leq |f(\lambda)| \leq |c| + r$. That is,

$$\frac{|f(z)| - |\phi_z(\lambda)|}{1 - |f(z)||\phi_z(\lambda)|} \leq |f(\lambda)| \leq \frac{|f(z)| + |\phi_z(\lambda)|}{1 + |f(z)||\phi_z(\lambda)|}. \tag{1.8}$$

(B) For $x \in [0,1]$, $(t + x)/(1 + tx)$ and $(t - x)/(1 - tx)$ are increasing functions of $t \in [0,1]$.

(C)

$$\left(1 + \frac{(y+x)/(1+yx) + x}{1 + ((y+x)/(1+yx))x}\right) \div \left(1 - \frac{(y+x)/(1+yx) + x}{1 + ((y+x)/(1+yx))x}\right) = \frac{1+y}{1-y}\left(\frac{1+x}{1-x}\right)^2. \tag{1.9}$$

(D)

$$\left(1 + \frac{(y-x)/(1-yx) - x}{1 - ((y-x)/(1-yx))x}\right) \div \left(1 - \frac{(y-x)/(1-yx) - x}{1 - ((y-x)/(1-yx))x}\right) = \frac{1+y}{1-y}\left(\frac{1-x}{1+x}\right)^2. \tag{1.10}$$

## 2. Results

We see below that the following has Theorem 1.3 as a consequence.

Proposition 2.1. *Let $f:\Delta\to\Delta$ be analytic. Then for all $z_1, z_2\in\Delta$,*

$$\frac{(|f^*(z_1)|-|\phi_{z_1}(z_2)|)/(1-|f^*(z_1)||\phi_{z_1}(z_2)|)-|\phi_{z_1}(z_2)|}{1-\big(|f^*(z_1)|-|\phi_{z_1}(z_2)|)/(1-|f^*(z_1)||\phi_{z_1}(z_2)|)|\phi_{z_1}(z_2)|} \le |f^*(z_2)| \le \frac{(|f^*(z_1)|+|\phi_{z_1}(z_2)|)/(1+|f^*(z_1)||\phi_{z_1}(z_2)|)+|\phi_{z_1}(z_2)|}{1+((|f^*(z_1)|+|\phi_{z_1}(z_2)|)/(1+|f^*(z_1)||\phi_{z_1}(z_2)|))|\phi_{z_1}(z_2)|}. \tag{2.1}$$

*Proof.* For $f:\Delta\to\Delta$ analytic, we fix $w_1=f(z_1)$, $w_2=f(z_2)$ and set

$$g=(\phi_{w_2}\circ f)/\phi_{z_2},\qquad h=(\phi_{w_1}\circ f)/\phi_{z_1}. \tag{2.2}$$

By Schwarz-Pick Lemma 1.2, we have $g,h:\Delta\to\Delta$, and

$$\begin{aligned} g(z_1)&=\frac{w_2-w_1}{z_2-z_1}\frac{1-\overline{z_2}z_1}{1-\overline{w_2}w_1}, \qquad g(z_2)=f^*(z_2),\\ h(z_2)&=\frac{w_2-w_1}{z_2-z_1}\frac{1-z_2\overline{z_1}}{1-w_2\overline{w_1}}, \qquad h(z_1)=f^*(z_1). \end{aligned} \tag{2.3}$$

The estimates in (A) give

$$\begin{aligned} \frac{|g(z_1)|-|\phi_{z_1}(z_2)|}{1-|g(z_1)||\phi_{z_1}(z_2)|}&\le|g(z_2)|\le\frac{|g(z_1)|+|\phi_{z_1}(z_2)|}{1+|g(z_1)||\phi_{z_1}(z_2)|},\\ \text{that is,}\quad \frac{|h(z_2)|-|\phi_{z_1}(z_2)|}{1-|h(z_2)||\phi_{z_1}(z_2)|}&\le|g(z_2)|\le\frac{|h(z_2)|+|\phi_{z_1}(z_2)|}{1+|h(z_2)||\phi_{z_1}(z_2)|}. \end{aligned} \tag{2.4}$$

Applying estimates (A) to $|h(z_2)|$ now (and observing (B)), we obtain the desired result. □

*Remark 2.2.* If $f$ is not an automorphism, then we may apply the increasing function $t\mapsto(1/2)\log((1+t)/(1-t))$ to either side of Proposition 2.1, and we use (C) and (D) to obtain

$$\rho(f^*(z_1),0)-2\rho(z_1,z_2)\le\rho(f^*(z_2),0)\le\rho(f^*(z_1),0)+2\rho(z_1,z_2), \tag{2.5}$$

which is Theorem 1.3.

A more careful analysis yields a little more. With the same notation, we set

$$\begin{aligned} \sigma_1=g(z_1)&=\frac{w_2-w_1}{z_2-z_1}\frac{1-\overline{z_2}z_1}{1-\overline{w_2}w_1},\\ \sigma_2=h(z_2)&=\frac{w_2-w_1}{z_2-z_1}\frac{1-z_2\overline{z_1}}{1-w_2\overline{w_1}}, \end{aligned} \tag{2.6}$$

$p = \phi_{f^*(z_1)} \circ g$, and $q = \phi_{\sigma_1} \circ h$. Here, estimates in (A) give

$$\frac{|p(z_1)| - |\phi_{z_1}(z_2)|}{1 - |p(z_1)||\phi_{z_1}(z_2)|} \le |p(z_2)| \le \frac{|p(z_1)| + |\phi_{z_1}(z_2)|}{1 + |p(z_1)||\phi_{z_1}(z_2)|}. \tag{2.7}$$

As before $|p(z_1)| = |q(z_1)|$, and applying (A) (and (B)) gives

$$\begin{aligned} |p(z_2)| &= |\phi_{f^*(z_1)}(f^*(z_2))| \\ &\le \frac{(|q(z_2)| + |\phi_{z_1}(z_2)|)/(1 + |q(z_2)||\phi_{z_1}(z_2)|) + |\phi_{z_1}(z_2)|}{1 + ((|q(z_2)| + |\phi_{z_1}(z_2)|)/(1 + |q(z_2)||\phi_{z_1}(z_2)|))|\phi_{z_1}(z_2)|} \\ &= \frac{(|\phi_{\sigma_1}(\sigma_2)| + |\phi_{z_1}(z_2)|)/(1 + |\phi_{\sigma_1}(\sigma_2)||\phi_{z_1}(z_2)|) + |\phi_{z_1}(z_2)|}{1 + ((|\phi_{\sigma_1}(\sigma_2)| + |\phi_{z_1}(z_2)|)/(1 + |\phi_{\sigma_1}(\sigma_2)||\phi_{z_1}(z_2)|))|\phi_{z_1}(z_2)|}. \end{aligned} \tag{2.8}$$

Likewise,

$$\frac{(|\phi_{\sigma_1}(\sigma_2)| - |\phi_{z_1}(z_2)|)/(1 - |\phi_{\sigma_1}(\sigma_2)||\phi_{z_1}(z_2)|) - |\phi_{z_1}(z_2)|}{1 - ((|\phi_{\sigma_1}(\sigma_2)| - |\phi_{z_1}(z_2)|)/(1 - |\phi_{\sigma_1}(\sigma_2)||\phi_{z_1}(z_2)|))|\phi_{z_1}(z_2)|} \le |\phi_{f^*(z_1)}(f^*(z_2))|. \tag{2.9}$$

Again applying the increasing function $t \mapsto (1/2)\log((1+t)/(1-t))$ when $f$ is not an automorphism, we obtain the following, which improves Theorem 1.4. (Having $z_2 = 0$ and requiring $f(0) = 0$ yield $\sigma_1 = \sigma_2$.)

Proposition 2.3. *For $f : \Delta \to \Delta$ analytic and not an automorphism,*

$$|\rho(f^*(z_1), f^*(z_2)) - \rho(\sigma_1, \sigma_2)| \le 2\rho(z_1, z_2) \quad \forall z_1, z_2 \in \Delta. \tag{2.10}$$

*Remark 2.4.* We cite [3], which contains various other generalizations of Theorem 1.4, one of which (Corollary 4.4) has conclusion

$$\rho\left(\frac{1 - z_1\overline{z_2}}{\overline{z_1}z_2 - 1} f^*(z_1), \frac{1 - w_1\overline{w_2}}{\overline{w_1}w_2 - 1} f^*(z_2)\right) \le 2\rho(z_1, z_2) \quad \forall z_1, z_2 \in \Delta. \tag{2.11}$$

([3] also contains some Euclidean versions, as does [5].)

## 3. Other results

Theorem 1.3 is obtained in [6] by integrating the following theorem.

Theorem 3.1 (see [6]). *Let $f : \Delta \to \Delta$ be analytic. Then,*

$$\left|\frac{d}{dz}|f^*(z)|\right| \le \frac{1 - |f^*(z)|^2}{1 - |z|^2}. \tag{3.1}$$

Below we refine this result using the same sort of procedure as above. (Then, in principle, a sharpening of Theorem 1.3 could be obtained via integration.)

Proposition 3.2. *Let $f:\Delta\to\Delta$ be analytic. Then,*

$$\left|\frac{d}{dz}\left|f^*(z)\right|\right| \le \frac{\left|\phi_{f^*(z)}\left(\phi_{f(z)}(f(0))/z\right)\right|+\left|z\right|^2}{\left|z\right|\left(1+\left|\phi_{f^*(z)}\left(\phi_{f(z)}(f(0))/z\right)\right|\right)}\,\frac{1-\left|f^*(z)\right|^2}{1-\left|z\right|^2}. \tag{3.2}$$

*Proof.* With $f$ as given, set

$$g(\lambda)=\phi_{f(z)}\circ\left(f\circ\phi_z(\lambda)\right), \qquad h(\lambda)=\phi_{g'(0)}\left(g(\lambda)/\lambda\right). \tag{3.3}$$

Then $g(0)=0$, and so $h(0)=0$. We apply the upper estimate in (A) to $h(\lambda)/\lambda$, then have $\lambda\to 0$, to obtain

$$\left|h'(0)\right| \le \frac{|h(z)|+|z|^2}{|z|(1+|h(z)|)}. \tag{3.4}$$

Now $h'(0)=g''(0)/2(|g'(0)|^2-1)$, and so

$$\frac{\left|g''(0)\right|}{2\left(1-\left|g'(0)\right|^2\right)} \le \frac{|h(z)|+|z|^2}{|z|(1+|h(z)|)}. \tag{3.5}$$

Here $g'(0)=f^*(z)$, and a straightforward computation (cf. [6, Section 2]) reveals that

$$\left|g''(0)\right| = 2\left(1-|z|^2\right)\left|\frac{d}{dz}\left|f^*(z)\right|\right|, \tag{3.6}$$

as desired. □

*Remarks 3.3.* (i) Schwarz's Lemma 1.1 applied to $h$ gives $(|\phi_{f^*(z)}(\phi_{f(z)}(f(0))/z)|+|z|^2)/|z|(1+|\phi_{f^*(z)}(\phi_{f(z)}(f(0))/z)|)\le 1$, so this is indeed a refinement. (ii) The lower estimate in (A) would similarly yield a lower estimate for $|d/dz|f^*(z)||$. We leave the details to the reader. (iii) In [6], the author compares Theorem 3.1 with Schwarz-Pick Lemma 1.2. Proposition 3.2 may be similarly compared with Dieudonné's lemma (e.g., [2, 4]), which refines Schwarz-Pick Lemma 1.2. A perfect analog of Dieudonné's lemma would read $|d/dz|f^*(z)|| \le ((|f^*(z)|+|z|^2)/|z|(1+|f^*(z)|))((1-|f^*(z)|^2)/(1-|z|^2))$ (for $f^*(0)=0$). However, this is not a refinement: for $f(\lambda)=\lambda^2$, we have $|d/dz|f^*(z)||=(1-|f^*(z)|^2)/(1-|z|^2)$ but $(|f^*(z)|+|z|^2)/|z|(1+|f^*(z)|)=2$ when $z=0$. (At any $z$ for which $f(z)=f(0)$, we have $|h(z)|=|f^*(z)|$, so a perfect analog does occur at such points.)

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