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Research Article

New Inequalities on Fractal Analysis and Their Applications

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Two new fractal measures M^{*s} and M_*^s are constructed from Minkowski contents M^{*s} and M_*^s. The properties of these two new measures are studied. We show that the fractal dimensions Dim and $\widehat{\delta}$ can be derived from M^{*s} and M_*^s, respectively. Moreover, some inequalities about the dimension of product sets and product measures are obtained.

1. Introduction

Hausdorff measure and packing measure are two of the most important fractal measures used in studying fractal sets (see [1–5]). They also yield Hausdorff dimension dim and packing dimension Dim, whose main properties are the following.

Property 1.1 (monotonicity). $E_1 \subset E_2 \Rightarrow \dim(E_1) \le \dim(E_2)$, $\mathrm{Dim}(E_1) \le \mathrm{Dim}(E_2)$.

Property 1.2 (σ-stability). $\dim(\bigcup_n E_n) \le \sup_n \dim(E_n)$, $\mathrm{Dim}(\bigcup_n E_n) \le \sup_n \mathrm{Dim}(E_n)$.

Not all dimension indices are σ-stable. For example, upper box dimension Δ and lower box dimension δ are not σ-stable. These two indices can be yielded from the upper and lower Minkowski contents M^{*s} and M_*^s. We know that the Minkowski contents are not outer measures as they are not countably subadditive. It is known that the modified upper box dimension $\widehat{\Delta}$ and the modified lower box dimension $\widehat{\delta}$ are dimension indices which satisfy Properties 1.1 and 1.2. However, until now no measures have been constructed that yield $\widehat{\Delta}$ and $\widehat{\delta}$. In the first part of this paper, we construct two Borel regular measures $\mathcal{M}^{*s}$ and $\mathcal{M}_*^s$. The properties of these two new measures, many of which mirror those of packing measure, are studied. We show that they yield $\widehat{\Delta}$ and $\widehat{\delta}$, respectively.

The first result about the Hausdorff dimension of the Cartesian product of sets in Euclidean space was obtained by Besicovitch and Moran [6]. Readers can also consult the book of Falconer [2] for a good survey. In [5], Tricot gives a complete description of Hausdorff and packing dimensions as follows:

$$\begin{aligned} \dim(E)+\dim(F) \le \dim(E\times F) &\le \dim(E)+\operatorname{Dim}(F) \\ &\le \operatorname{Dim}(E\times F) \le \operatorname{Dim}(E)+\operatorname{Dim}(F). \end{aligned} \tag{1.1}$$

Connecting to $\widehat{\delta}$, Xiao [7] proves the following result:

$$\widehat{\delta}(E)+\operatorname{Dim}(F) \le \operatorname{Dim}(E\times F). \tag{1.2}$$

In this paper, we first prove the following inequality:

$$\widehat{\delta}(E)+\widehat{\delta}(F) \le \widehat{\delta}(E\times F) \le \widehat{\delta}(E)+\operatorname{Dim}(F). \tag{1.3}$$

As a consequence, we have the following inequality:

$$\widehat{\delta}(E)+\widehat{\delta}(F) \le \widehat{\delta}(E\times F) \le \widehat{\delta}(E)+\operatorname{Dim}(F) \le \operatorname{Dim}(E\times F) \le \operatorname{Dim}(E)+\operatorname{Dim}(F). \tag{1.4}$$

We also show that the inequality $\dim(E\times F) \le \dim(E)+\widehat{\delta}(F)$ does not hold.

On the other hand, Haase [8] studies the dimension of product measures and obtains the following result:

$$\begin{aligned} \dim(\mu)+\dim(\nu) \le \dim(\mu\times\nu) &\le \dim(\mu)+\operatorname{Dim}(\nu) \\ &\le \operatorname{Dim}(\mu\times\nu) \le \operatorname{Dim}(\mu)+\operatorname{Dim}(\nu). \end{aligned} \tag{1.5}$$

Using the properties of $\mathcal{M}^s_*$, here we prove a new inequality as follows:

$$\widehat{\delta}(\mu)+\widehat{\delta}(\nu) \le \widehat{\delta}(\mu\times\nu) \le \widehat{\delta}(\mu)+\operatorname{Dim}(\nu) \le \operatorname{Dim}(\mu\times\nu) \le \operatorname{Dim}(\mu)+\operatorname{Dim}(\nu). \tag{1.6}$$

2. Background

Let us first recall some basic properties of Hausdorff measure, Hausdorff dimension, packing measure, packing dimension, Minkowski contents, box dimensions, and modified box dimensions.

Let U be a nonempty subset of $\mathbb{R}^n$. As usual, one may define the diameter of U as

$$|U| = \sup\{|x-y| : x,y \in U\}. \tag{2.1}$$

Let E be a subset of $\mathbb{R}^n$ and $s>0$. For $\delta>0$, define

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} |E_i|^s : E \subset \bigcup_i E_i,\ |E_i| \le \delta\right\}. \tag{2.2}$$

It is easy to check that $\mathcal{H}^s_\delta$ is an outer measure on $\mathbb{R}^n$.

We define the s-dimensional Hausdorff measure of E by

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E). \tag{2.3}$$

It is known that $\mathcal{H}^s$ is a Borel regular measure (see Stein and Shakarchi [9, Chapter 7]). The Hausdorff dimension of E can be defined as

$$\dim(E) = \inf\{s > 0 : \mathcal{H}^s(E) = 0\} = \sup\{s > 0 : \mathcal{H}^s(E) > 0\}. \tag{2.4}$$

Define

$$P^s_\delta(E) = \sup\left\{\sum_{i=1}^{\infty} |2r_i|^s : B(x_i, r_i)s \text{ are pairwisely disjoint, } x_i \in E,\ r_i < \delta\right\}, \tag{2.5}$$

where $B(x,r)$ is the closed ball centered at x with radius r. Then the premeasure $P^s(E)$ of E is defined as (see Tricot [5])

$$P^s(E) = \lim_{\delta\to 0} P^s_\delta(E). \tag{2.6}$$

It is known that $P^s(E)$ is not an outer measure since it fails to be countably subadditive. However, the s-dimensional packing measure of E, which is a Borel regular measure, can be defined as

$$\mathcal{P}^s(E) = \inf\left\{\sum_{i=1}^{\infty} P^s(E_i) : E \subset \bigcup_i E_i\right\}. \tag{2.7}$$

The packing dimension of E is defined by

$$\operatorname{Dim}(E) = \inf\{s > 0 : \mathcal{P}^s(E) = 0\} = \sup\{s > 0 : \mathcal{P}^s(E) > 0\}. \tag{2.8}$$

If E is a bounded subset in $\mathbb{R}^n$, for $\varepsilon > 0$, denote

$$E(\varepsilon) = \{x \in \mathbb{R}^n : d(x, E) \le \varepsilon\}, \tag{2.9}$$

which is called a closed ε-neighborhood of E. Associating to ε, one may also define the covering number

$$N(E, \varepsilon) = \min\left\{k : E \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon)\right\}, \tag{2.10}$$

and the packing number

$$P(E, \varepsilon) = \max\{k : \text{there are disjoint balls } B(x_i, \varepsilon),\ i = 1, \ldots, k,\ x_i \in E\}. \tag{2.11}$$

The s-dimensional upper and lower Minkowski contents of bounded set E are defined by

$$M^{*s}(E) = \limsup_{\varepsilon\downarrow 0}\{(2\varepsilon)^{s-n}\mathscr{L}^n(E(\varepsilon))\},$$
$$M_*^s(E) = \liminf_{\varepsilon\downarrow 0}\{(2\varepsilon)^{s-n}\mathscr{L}^n(E(\varepsilon))\}, \tag{2.12}$$

where $\varepsilon\downarrow 0$ and $\mathscr{L}^n$ are the Lebesgue measures on $\mathbb{R}^n$.

Thus we can define the upper and lower box dimensions by

$$\Delta(E) = \inf\{s : M^{*s}(E) = 0\} = \sup\{s : M^{*s}(E) > 0\},$$
$$\delta(E) = \inf\{s : M_*^s(E) = 0\} = \sup\{s : M_*^s(E) > 0\}. \tag{2.13}$$

It is known that Minkowski contents are not outer measures as they are not countable subadditive, and the indices Δ,δ are not σ-stable (see, e.g., Tricot [5], Falconer [1]). We can obtain σ-stable indices $\widehat{\Delta}$ and $\widehat{\delta}$, which are called the modified upper and lower box dimensions, by letting

$$\widehat{\Delta}(E) = \inf\left\{\sup_i \Delta(E_i) : E\subset\bigcup_i E_i,\ E_is \text{ are bounded}\right\},$$
$$\widehat{\delta}(E) = \inf\left\{\sup_i \delta(E_i) : E\subset\bigcup_i E_i,\ E_is \text{ are bounded}\right\}. \tag{2.14}$$

In [5], Tricot proves that $\mathrm{Dim} = \widehat{\Delta}$, and Falconer [1] shows that for any set $E\subset\mathbb{R}^n$,

$$0 \le \dim(E) \le \widehat{\delta}(E) \le \widehat{\Delta}(E) = \mathrm{Dim}(E) \le n. \tag{2.15}$$

In order to prove the results in this paper, the following two auxiliary lemmas are needed, which can be found by Mattila in [3, Lemmas 5.4 and 5.5].

Lemma 2.1. *$N(E,2\varepsilon) \le P(E,\varepsilon) \le N(E,\varepsilon/2)$ for any subset E of $\mathbb{R}^n$.*

Lemma 2.2. *$P(E,\varepsilon)a_n\varepsilon^n \le \mathscr{L}^n(E(\varepsilon)) \le N(E,\varepsilon)a_n(2\varepsilon)^n$, where $a_n = \mathscr{L}^n(B(0,1))$.*

The following lemma is from [1, Example 7.8].

Lemma 2.3. *There exist sets $E,F\subset\mathbb{R}$ with $\delta(E) = \delta(F) = 0$ and $\dim(E\times F)\ge 1$.*

For reader's convenience, we give the example as follows.

Let $0 = m_0 < m_1 < \cdots$ be a rapidly increasing sequence of integers satisfying a condition to be specified below. Let E be a set of real numbers in $[0,1]$ with zero in the rth decimal place whenever $m_k + 1 \le r \le m_{k+1}$ with $k = 2\ell$, $\ell\in\mathbb{Z}_+$. Similarly, let F be a set of real numbers with zero in the rth decimal place if $m_k + 1 \le r \le m_{k+1}$ with $k = 2\ell+1$, $\ell\in\mathbb{Z}_+$. Looking at the first m_{k+1} decimal places for even k, there is an obvious cover of E by 10^{j_k} intervals of length $10^{-m_{k+1}}$, where

$$j_k = (m_2 - m_1) + (m_4 - m_3) + \cdots + (m_k - m_{k-1}). \tag{2.16}$$

Then $\log 10^{j_k}/-\log 10^{-m_{k+1}} = j_k/m_{k+1}$ which tends to 0 as $k \to \infty$ provided that the m_k are chosen to increase sufficiently rapidly. So we have $\delta(E) = 0$. Similarly, $\delta(F) = 0$.

If $0 < w < 1$, then we can write $w = x + y$, where $x \in E$ and $y \in F$; just take the rth decimal digit of w from E if $m_k + 1 \le r \le m_{k+1}$ and k is odd and from F if k is even. The mapping $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x + y$ is easily seen to be Lipschitz, so

$$\dim(E \times F) \ge \dim f(E \times F) \ge \dim\big((0,1)\big) = 1 \tag{2.17}$$

by [1, Corollary 2.4(a)].

The following lemma summarizes some of the basic properties of Minkowski contents.

Lemma 2.4. *Let M^s be one of M^{*s} and M^s_*, then for bounded sets E, F, $\{E_i\}$,*

(i) *$M^s(\varnothing) = 0$;*

(ii) *M^s is monotone: $E_1 \subset E_2 \Rightarrow M^s(E_1) \le M^s(E_2)$;*

(iii) *$M^s(E) = M^s(\overline{E})$;*

(iv) *assume that $s < t$. If $M^s(E) < \infty$, then $M^t(E) = 0$. Moreover, if $M^t(E) > 0$, then $M^s(E) = \infty$;*

(v) *$M^{*s}(E \cup F) \le M^{*s}(E) + M^{*s}(F)$, $M^s_*(\bigcup_i E_i) \ge \sum_i M^s_*(E_i)$ for $d(E_i, E_j) > c > 0$, $i \ne j$;*

(vi) *if $E = \{x\}$, then $M^0(E) = 2^{-n}a_n$, $M^s(E) = 0$, $s > 0$;*

(vii) *if $0 < \mathscr{L}^n(E) < \infty$, then $M^n(E) = \mathscr{L}^n(E)$, $M^s(E) = \infty$, $s < n$.*

Proof. (i), (ii) are trivial. (iii) follows from $E(\varepsilon) = \overline{E}(\varepsilon)$. (iv) derives from the equality

$$(2\varepsilon)^{s-n}\mathscr{L}^n\big(E(\varepsilon)\big) = (2\varepsilon)^{s-t}(2\varepsilon)^{t-n}\mathscr{L}^n\big(E(\varepsilon)\big). \tag{2.18}$$

(v) The first inequality is obvious.

We have $\mathrm{d}(E_i(\varepsilon), E_j(\varepsilon)) > 0$ for $i \ne j$ when $0 < 2\varepsilon < c$, thus

$$\begin{aligned} M^s_*\Big(\bigcup_i E_i\Big) &= \liminf_{\varepsilon\downarrow 0}\Big\{(2\varepsilon)^{s-n}\mathscr{L}^n\Big(\Big(\bigcup_i E_i\Big)(\varepsilon)\Big)\Big\} \\ &= \liminf_{\varepsilon\downarrow 0}\Big\{(2\varepsilon)^{s-n}\sum_i \mathscr{L}^n\big(E_i(\varepsilon)\big)\Big\} \\ &\ge \sum_i \liminf_{\varepsilon\downarrow 0}\big\{(2\varepsilon)^{s-n}\mathscr{L}^n\big(E_i(\varepsilon)\big)\big\} = \sum_i M^s_*(E_i). \end{aligned} \tag{2.19}$$

(vi) Follows from $(2\varepsilon)^{s-n}\mathscr{L}^n(x(\varepsilon)) = a_n(2\varepsilon)^{s-n}\varepsilon^n = 2^{s-n}a_n\varepsilon^s$.

(vii) Holds since

$$\begin{gathered} \lim_{\varepsilon\downarrow 0}(2\varepsilon)^{n-n}\mathscr{L}^n\big(E(\varepsilon)\big) = \mathscr{L}^n(E), \\ \lim_{\varepsilon\downarrow 0}\big\{(2\varepsilon)^{s-n}\mathscr{L}^n\big(E(\varepsilon)\big)\big\} \ge \lim_{\varepsilon\downarrow 0}(2\varepsilon)^{s-n}\mathscr{L}^n(E) = \infty \quad \text{for } s < n. \end{gathered} \tag{2.20}$$

□

3. The dimensions of product sets

In this section, we give a formula about dimensions of product sets. First let us state a lemma from Bishop and Peres [10, Lemma 2.1].

Lemma 3.1. *Let E be a subset of a separable metric space, with $\hat{\delta}(E) > \alpha$ (or $\mathrm{Dim}(E) > \alpha$). Then there is a (relatively closed) nonempty subset F of E, such that $\hat{\delta}(F \cap V) > \alpha$ (or $\mathrm{Dim}(F \cap V) > \alpha$) for any open set V which intersects F.*

Theorem 3.2. *For any subsets E, F of $\mathbb{R}^n$,*

$$\hat{\delta}(E) + \hat{\delta}(F) \le \hat{\delta}(E \times F) \le \hat{\delta}(E) + \mathrm{Dim}(F) \le \mathrm{Dim}(E \times F) \le \mathrm{Dim}(E) + \mathrm{Dim}(F). \tag{3.1}$$

Proof. (i) First we prove the first inequality. Here we modify the proof of Theorem 4.1 in [7], where E is Borel set and F is compact.

It suffices to show that

$$\hat{\delta}(E \times F) \ge \alpha + \beta \tag{3.2}$$

for any $\alpha < \hat{\delta}(E)$, $\beta < \hat{\delta}(F)$.

By Lemma 3.1, there exist closed sets $E_\alpha \subset E$, $F_\beta \subset F$ such that

$$\hat{\delta}(E_\alpha \cap V) > \alpha, \qquad \hat{\delta}(F_\beta \cap W) > \beta \tag{3.3}$$

for any open sets V, W, where $V \cap E_\alpha \neq \emptyset, W \cap F_\beta \neq \emptyset$.

For any $\varepsilon > 0$, we may find bounded $\{G_n\}$ with $E_\alpha \times F_\beta \subset \bigcup_n G_n$, and for any n,

$$\delta(G_n) \le \hat{\delta}(E_\alpha \times F_\beta) + \varepsilon \le \hat{\delta}(E \times F) + \varepsilon. \tag{3.4}$$

Since $\delta(G_n) = \delta(\overline{G_n})$, we may take G_n to be closed and $G_n \cap (E_\alpha \times F_\beta) \neq \emptyset$. By Baire's category theorem, we know that there exist n and an open set U which intersects $E_\alpha \times F_\beta$ such that $U \cap (E_\alpha \times F_\beta) \subset G_n$. Therefore, we may find open sets V, W such that $V \times W \subset U$ and $(V \times W) \cap (E_\alpha \times F_\beta) \neq \emptyset$, then we have

$$(E_\alpha \cap V) \times (F_\beta \cap W) \subset G_n, \tag{3.5}$$

hence

$$\begin{aligned} \alpha + \beta &\le \hat{\delta}(E_\alpha \cap V) + \hat{\delta}(F_\beta \cap W) \\ &\le \delta(E_\alpha \cap V) + \delta(F_\beta \cap W) \\ &\le \delta((E_\alpha \cap V) \times (F_\beta \cap W)) \\ &\le \delta(G_n) \le \hat{\delta}(E \times F) + \varepsilon, \end{aligned} \tag{3.6}$$

the third inequality follows from the definitions of the upper and lower box dimensions. Since ε is arbitrary, (3.2) follows immediately.

(ii) Now let us turn to the second inequality. Suppose $E \subset \bigcup_i E_i$, $F \subset \bigcup_i F_i$, E_is and F_is are bounded, then $E \times F \subset \bigcup_{i,j}(E_i \times F_i)$, thus

$$\begin{aligned}
\widehat{\delta}(E\times F) &= \inf_{E\times F\subset\bigcup_l V_l}\left\{\sup_l \delta(V_l) : E\times F\subset\bigcup_l V_l, V_l s \text{ are bounded}\right\}\\
&\le \inf_{E\times F\subset\bigcup_{i,j}(E_i\times F_j)}\left\{\sup_{i,j}\delta(E_i\times F_j) : E\times F\subset\bigcup_{i,j}(E_i\times F_j)\right\}\\
&\le \inf\left\{\sup_{i,j}\left(\delta(E_i)+\Delta(F_j)\right) : E\subset\bigcup_i E_i,\ F\subset\bigcup_j F_j\right\}\\
&\le \inf\left\{\sup_i \delta(E_i) : E\subset\bigcup_i E_i\right\}+\inf\left\{\sup_j \Delta(F_j) : F\subset\bigcup_j F_j\right\}\\
&= \widehat{\delta}(E)+\widehat{\Delta}(F),
\end{aligned} \tag{3.7}$$

the second inequality above follows from the definitions of the upper and lower box dimensions.

(iii) The proof of the third inequality is similar to (i).

(iv) The last one can be referred to Tricot [5, Theorem 3]. □

Remark 3.3. (a) One may ask whether $\dim(E\times F) \le \dim(E) + \widehat{\delta}(F)$ holds or not. By Lemma 2.3, we know that there exist sets $E,F \subset \mathbb{R}$ with $\delta(E) = \delta(F) = 0$ and $\dim(E \times F) \ge 1$. Hence,

$$\dim(E\times F) > \delta(E)+\delta(F) \ge \dim(E)+\delta(F) \ge \dim(E)+\widehat{\delta}(F). \tag{3.8}$$

(b) As a consequence of (2.15) and Theorem 3.2, one has

$$\begin{aligned}
\dim(E)+\dim(F) &\le \dim(E\times F)\le \widehat{\delta}(E\times F)\le \widehat{\delta}(E)+\operatorname{Dim}(F)\\
&\le \operatorname{Dim}(E\times F)\le \operatorname{Dim}(E)+\operatorname{Dim}(F).
\end{aligned} \tag{3.9}$$

4. $\mathcal{M}^{*s}$, $\mathcal{M}_*^s$, and their dimensions D, d

It is known that the Minkowski contents are not outer measures since they fail to be countably subadditive. In fact, we may derive this assertion directly from Lemma 2.4. Consider $s = 1$ and $E = \mathbb{Q} \cap [0,1]$, the set of rational numbers in $[0,1]$. By Lemma 2.4, we know that $M^1(E) = M^1([0,1]) = 1$ and $M^1(\{q\}) = 0$ for any $q \in E$, thus $\sum_{q\in E} M^1(\{q\}) = 0$.

We use a standard procedure and define

$$\begin{aligned}
\mathcal{M}^{*s}(E) &= \inf\left\{\sum_{i=1}^{\infty} M^{*s}(E_i) : E=\bigcup_i E_i,\ E_i s \text{ are bounded}\right\},\\
\mathcal{M}_*^{s}(E) &= \inf\left\{\sum_{i=1}^{\infty} M_*^{s}(E_i) : E=\bigcup_i E_i,\ E_i s \text{ are bounded}\right\}.
\end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $\mathcal{M}^s$ be one of $\mathcal{M}^{*s}$ and $\mathcal{M}^s_*$, then*

(i) *$\mathcal{M}^s$ is an outer measure;*
(ii) *$\mathcal{M}^s$ is metric: $d(E,F) > 0 \Rightarrow \mathcal{M}^s(E \cup F) = \mathcal{M}^s(E) + \mathcal{M}^s(F)$;*
(iii) *$\mathcal{M}^s$ is a Borel measure;*
(iv) *$\mathcal{M}^s$ is Borel regular: for all $E \subset \mathbb{R}^n$, there is a Borel set $B \supset E$ such that $\mathcal{M}^s(B) = \mathcal{M}^s(E)$;*
(v) *$\mathcal{M}^s(E) \le M^s(E)$ for bounded set E;*
(vi) *$\mathcal{M}^s(E_n) \to \mathcal{M}^s(E)$ for any sequence of sets $E_n \uparrow E$;*
(vii) *if E is $\mathcal{M}^s$-measurable, $0 < \mathcal{M}^s(E) < \infty$, and $\varepsilon > 0$, there exists a closed set $F \subset E$ such that $\mathcal{M}^s(F) > \mathcal{M}^s(E) - \varepsilon$;*
(viii) *for any E,*

$$\mathcal{M}^{*s}(E) = \inf\left\{ \lim_{n\to\infty} M^{*s}(E_n) : E_n \uparrow E,\ E_n s \text{ are bounded}\right\}. \tag{4.2}$$

Proof. Let M^s be one of M^{*s} and M^s_*.

(i) $\mathcal{M}^s(\varnothing) = 0$ and that $\mathcal{M}^s$ is monotone are obvious, so it suffices to verify that $\mathcal{M}^s$ is countably subadditive. Suppose that $E = \bigcup_i E_i$, for any $\varepsilon > 0$, there exist bounded sets $\{E_{ij}\}$ such that $E_i = \bigcup_j E_{ij}$, $\sum_j M^s(E_{ij}) < \mathcal{M}^s(E_i) + \varepsilon/2^i$, thus

$$\begin{gathered} E = \bigcup_i E_i = \bigcup_i \bigcup_j E_{ij}, \\ \mathcal{M}^s(E) \le \sum_i \sum_j M^s(E_{ij}) \le \sum_i \left(\mathcal{M}^s(E_i) + \frac{\varepsilon}{2^i}\right) = \sum_i \mathcal{M}^s(E_i) + \varepsilon. \end{gathered} \tag{4.3}$$

So we have $\mathcal{M}^s(E) \le \sum_i \mathcal{M}^s(E_i)$ by the arbitrariness of ε.

(ii) Assume that $E \cup F = \sum_i A_i$, A_is are bounded, then

$$\sum_i M^s(A_i) = \sum_{E\cap A_i \ne \varnothing} M^s(A_i) + \sum_{F\cap A_i \ne \varnothing} M^s(A_i), \tag{4.4}$$

thus

$$\inf \sum_i M^s(A_i) \ge \inf \sum_{E\cap A_i \ne \varnothing} M^s(A_i) + \inf \sum_{F\cap A_i \ne \varnothing} M^s(A_i), \tag{4.5}$$

so we have

$$\mathcal{M}^s(E \cup F) \ge \mathcal{M}^s(E) + \mathcal{M}^s(F), \tag{4.6}$$

the opposite inequality holds since $\mathcal{M}^s$ is an outer measure by (i).

(iii) Follows from (ii) by Falconer [2, Theorem 1.5].

(iv) We have $M^s(E) = M^s(\overline{E})$ by (iii) of Lemma 2.4, thus

$$\mathcal{M}^s(E) = \inf\left\{ \sum_{i=1}^{\infty} M^s(B_i) : E \subset \bigcup_i B_i,\ B_i s \text{ are closed and bounded}\right\}. \tag{4.7}$$

For $i = 1, 2, \ldots$, choose closed sets $B_{i1}, B_{i2}, \ldots$, such that

$$E \subset \bigcup_j B_{ij}, \qquad \sum_{j=1}^{\infty} M^s(B_{ij}) \leq \mathcal{M}^s(E) + \frac{1}{i}. \tag{4.8}$$

Then $B = \bigcap_i \bigcup_j B_{ij}$ is a Borel set such that $E \subset B$ and $\mathcal{M}^s(E) = \mathcal{M}^s(B)$.

(v) Is obvious by the definition of $\mathcal{M}^s$.

(vi) Since $E_n \uparrow E$, we know that $\lim \mathcal{M}^s(E_n)$ exists and is $\leq \mathcal{M}^s(E)$ by the monotonicity of $\mathcal{M}^s$. By (iv), there exists Borel set $F_i \supset E_i$ with $\mathcal{M}^s(F_i) = \mathcal{M}^s(E_i)$, that is, $\mathcal{M}^s(F_i \backslash E_i) = 0$. Let

$$B_n = \bigcup_{i=1}^{n} F_i, \qquad B = \bigcup_n B_n, \tag{4.9}$$

then B_ns are Borel sets with $B_n \uparrow B$, $E_n \subset B_n$. Furthermore, we have

$$\begin{aligned} \mathcal{M}^s(B_n) &= \mathcal{M}^s\left(\bigcup_{i=1}^{n} F_i\right) = \mathcal{M}^s(F_n) + \mathcal{M}^s\left(\left(\bigcup_{i=1}^{n-1} F_i\right) \Big\backslash F_n\right) \\ &\leq \mathcal{M}^s(E_n) + \sum_{i=1}^{n-1} \mathcal{M}^s(F_i\ E_n) \\ &\leq \mathcal{M}^s(E_n) + \sum_{i=1}^{n-1} \mathcal{M}^s(F_i \backslash E_i) = \mathcal{M}^s(E_n), \end{aligned} \tag{4.10}$$

hence

$$\mathcal{M}^s(E) \geq \lim_{n\to\infty} \mathcal{M}^s(E_n) = \lim_{n\to\infty} \mathcal{M}^s(B_n) = \mathcal{M}^s(B) \geq \mathcal{M}^s(E) \tag{4.11}$$

by the fact that

$$E = \bigcup_n E_n \subset \bigcup_n B_n = B. \tag{4.12}$$

(vii) Let E be $\mathcal{M}^s$-measurable, then there exists a Borel set $B \supset E$ with $\mathcal{M}^s(B) = \mathcal{M}^s(E)$, that is, $\mathcal{M}^s(B \backslash E) = 0$. We can find a Borel set $B_1 \supset (B \backslash E)$ with $\mathcal{M}^s(B_1) = 0$, then $B_2 = B \backslash B_1$ is Borel, $B_2 \subset E$, and $\mathcal{M}^s(B_2) = \mathcal{M}^s(E)$. By [3, Theorem 1.9 and Corollary 1.11], we know that $\mathcal{M}^s | B_2$, the restriction of measure $\mathcal{M}^s$ to B_2, is a Radon measure, thus is an inner regular measure since $0 < \mathcal{M}^s(E) = \mathcal{M}^s(B_2) < \infty$, so there exists a closed set $F \subset B_2$ such that $\mathcal{M}^s|_{B_2}(F) > \mathcal{M}^s|_{B_2}(B_2) - \varepsilon$ which gives $\mathcal{M}^s(F) > \mathcal{M}^s(B_2) - \varepsilon = \mathcal{M}^s(E) - \varepsilon$.

(viii) The proof is the same as that of [4, Lemma 5.1(vii)]. □

Corollary 4.2. *For any subset E of $\mathbb{R}^n$,*

$$\mathcal{M}^s(E) = \inf\left\{\sum_{i=1}^{\infty} M^s(E_i) : E \subset \bigcup_i E_i,\ E_i s \textit{ are bounded Borel sets}\right\}. \tag{4.13}$$

Proof. We denote the right-hand side of the above equality by $\mu(E)$, then $\mathcal{M}^s(E) \leq \mu(E)$ follows from the definition of $\mathcal{M}^s(E)$ and $\mathcal{M}^s(E) \geq \mu(E)$ follows from (4.7). □

Corollary 4.3. *Let B be Borel set of $\mathbb{R}^n$, then*

$$\mathcal{M}^s(B)=\inf\left\{\sum_{i=1}^{\infty}M^s(B_i):B=\bigcup_i B_i,\ B_i s \text{ are disjoint bounded Borel sets}\right\}. \tag{4.14}$$

Proof. From (4.7), we have

$$\mathcal{M}^s(B)=\inf\left\{\sum_{i=1}^{\infty}M^s(F_i):B\subset\bigcup_i F_i,\ F_i s \text{ are closed and bounded}\right\}, \tag{4.15}$$

then $E_i=F_i\cap B$ is a bounded Borel set and $B=\bigcup_i E_i$. Take

$$B_1=E_1,\ B_2=E_2\backslash B_1,\dots,B_n=E_n\bigg\backslash\left(\bigcup_{i=1}^{n-1}B_i\right),\dots, \tag{4.16}$$

then $\{B_i\}$ are disjoint bounded Borel sets and $B=\bigcup_i B_i$, so we have

$$\mathcal{M}^s(B)\geq\inf\left\{\sum_{i=1}^{\infty}M^s(B_i):B=\bigcup_i B_i,\ B_i s \text{ are disjoint bounded Borel sets}\right\} \tag{4.17}$$

by the fact that $B_i\subset F_i$.

The opposite inequality holds by the definition of $\mathcal{M}^s$. □

Theorem 4.4. *For any subset E of $\mathbb{R}^n$, the following inequality holds:*

$$2^{-s-n}a_n\mathcal{H}^s(E)\leq\mathcal{M}_*^s(E)\leq\mathcal{M}^{*s}(E)\leq 2^s a_n\mathcal{P}^s(E). \tag{4.18}$$

Proof. The assertion $\mathcal{M}_*^s(E)\leq\mathcal{M}^{*s}(E)$ is trivial. We first prove the right-hand inequality, by Lemmas 2.1 and 2.2, for all bounded set $B\subset\mathbb{R}^n$,

$$\begin{aligned}M^{*s}(B)&=\limsup_{\varepsilon\downarrow 0}\{(2\varepsilon)^{s-n}\mathcal{L}^n(B(\varepsilon))\}\\&\leq\limsup_{\varepsilon\downarrow 0}\{(2\varepsilon)^{s-n}N(B,\varepsilon)a_n(2\varepsilon)^n\}\\&\leq\limsup_{\varepsilon\downarrow 0}\left\{2^s a_n P\left(B,\frac{\varepsilon}{2}\right)\varepsilon^s\right\}\\&\leq 2^s a_n\limsup_{\varepsilon\downarrow 0}P_\varepsilon^s(B)\leq 2^s a_n P^s(B),\end{aligned} \tag{4.19}$$

thus

$$\begin{aligned}\mathcal{M}^{*s}(E)&=\inf\left\{\sum_{i=1}^{\infty}M^{*s}(E_i):E=\bigcup_i E_i,\ E_i s \text{ are bounded}\right\}\\&\leq\inf\left\{\sum_{i=1}^{\infty}2^s a_n P^s(E_i):E=\bigcup_i E_i,\ E_i s \text{ are bounded}\right\}\\&=2^s a_n\mathcal{P}^s(B).\end{aligned} \tag{4.20}$$

The following is the proof of the left-hand side of the inequality. By Lemmas 2.1 and 2.2, we have for any bounded subset $B \subset \mathbb{R}^n$,

$$\begin{aligned} M_*^s(B) &= \liminf_{\varepsilon \downarrow 0} \{(2\varepsilon)^{s-n}\mathcal{L}^n(B(\varepsilon))\} \\ &\geq \liminf_{\varepsilon \downarrow 0} \{(2\varepsilon)^{s-n}P(B,\varepsilon)a_n\varepsilon^n\} \\ &\geq \liminf_{\varepsilon \downarrow 0} \{2^{s-n}a_nN(B,2\varepsilon)\varepsilon^s\} \\ &= 2^{-n-s}a_n \liminf_{\varepsilon \downarrow 0} \{N(B,2\varepsilon)(4\varepsilon)^s\} \\ &\geq 2^{-n-s}a_n \liminf_{\varepsilon \downarrow 0} \mathcal{H}_{4\varepsilon}^s(B) = 2^{-n-s}a_n\mathcal{H}^s(B). \end{aligned} \tag{4.21}$$

There exists a Borel set F such that $E \subset F, \mathcal{M}_*^s(E) = \mathcal{M}_*^s(F)$ since $\mathcal{M}_*^s$ is Borel regular. By Corollary 4.3, we have

$$\begin{aligned} \mathcal{M}_*^s(F) &= \inf\left\{\sum_{i=1}^{\infty} M_*^s(F_i) : F = \bigcup_i F_i,\ F_i s \text{ are disjoint bounded Borel sets}\right\} \\ &\geq 2^{-n-s}a_n \inf\left\{\sum_{i=1}^{\infty} \mathcal{H}^s(F_i) : F = \bigcup_i F_i,\ F_i s \text{ are disjoint bounded Borel sets}\right\} \\ &= 2^{-n-s}a_n\mathcal{H}^s(F) \geq 2^{-n-s}a_n\mathcal{H}^s(E). \end{aligned} \tag{4.22}$$

We complete the proof of the theorem. □

From Theorem 4.4 and its proof, we have the following corollary.

Corollary 4.5. *For any bounded subset E of $\mathbb{R}^n$, one has*

$$2^{-s-n}a_n\mathcal{H}^s(E) \leq \mathcal{M}_*^s(E) \leq M_*^s(E) \leq M^{*s}(E) \leq 2^s a_n P^s(E). \tag{4.23}$$

Now we can define two fractal dimensions from $\mathcal{M}^{*s}$ and $\mathcal{M}_*^s$ as follows:

$$\begin{aligned} d(E) &= \inf\{s : \mathcal{M}_*^s(E) = 0\} = \sup\{s : \mathcal{M}_*^s(E) = \infty\}, \\ D(E) &= \inf\{s : \mathcal{M}^{*s}(E) = 0\} = \sup\{s : \mathcal{M}^{*s}(E) = \infty\}. \end{aligned} \tag{4.24}$$

Thus by Theorem 4.4 and Corollary 4.5, we have

$$\begin{gathered} \dim(E) \leq d(E) \leq D(E) \leq \operatorname{Dim}(E) \leq \Delta(E), \\ \dim(E) \leq d(E) \leq \delta(E) \leq \Delta(E). \end{gathered} \tag{4.25}$$

In fact, we have the following formulas.

Theorem 4.6. *For any subset E of $\mathbb{R}^n$,*

(1) $D(E) = \operatorname{Dim}(E)$,

(2) $d(E) = \widehat{\delta}(E)$.

Proof. (1) It suffices to prove $\mathrm{D}(E) \geq \mathrm{Dim}(E)$. If $\mathrm{Dim}(E) > t$, $E = \bigcup_i E_i$, E_is are bounded, then $\sup_i \Delta(E_i) > t$ by the equivalent definition of $\mathrm{Dim}(E)$ as follows:

$$\mathrm{Dim}(E) = \inf\left\{\sup_i \Delta(E_i) : E \subset \bigcup_i E_i,\ E_i s \text{ are bounded}\right\}. \tag{4.26}$$

So there exists an i_0 such that $\Delta(E_{i_0}) > t$, then $M^{*t}(E_{i_0}) = \infty$ which implies that $\mathcal{M}^{*t}(E) = \infty$, so we have $\mathrm{D}(E) \geq t$, thus $\mathrm{D}(E) \geq \mathrm{Dim}(E)$.

(2) The proof of $\mathrm{d}(E) \geq \widehat{\delta}(E)$ is the same as that of (1). It suffices to prove $\widehat{\delta}(E) \geq d(E)$. If $t > \widehat{\delta}(E)$, then there exist bounded sets $\{E_i\}$ such that $E = \bigcup_i E_i$ and $t > \sup_i \delta(E_i) \geq \delta(E_i)$ for any i by the definition of $\widehat{\delta}$ as follows:

$$\widehat{\delta}(E) = \inf\left\{\sup_i \delta(E_i) : E \subset \bigcup_i E_i,\ E_i s \text{ are bounded}\right\}. \tag{4.27}$$

So we have $M_*^t(E_i) = 0$ for any i, thus $\mathcal{M}_*^t(E) \leq \sum_{i=1}^{\infty} M_*^t(E_i) = 0$ which implies that $\mathrm{d}(E) \leq t$, then we have $\widehat{\delta}(E) \geq \mathrm{d}(E)$. □

5. The dimensions of product measures

Let μ, ν be Borel probability measures on $\mathbb{R}^n$, $\mu \times \nu$ denotes the unique product measure. If α denotes any dimension index for a set, then for a measure μ, the corresponding dimension index $\alpha(\mu)$ is defined by

$$\alpha(\mu) = \inf\{\alpha(E) : \mu(E) > 0,\ E \text{ is a Borel set}\}. \tag{5.1}$$

From the above definition, we have

$$0 \leq \dim(\mu) \leq \widehat{\delta}(\mu) \leq \widehat{\Delta}(\mu) = \mathrm{Dim}(\mu) \leq n \tag{5.2}$$

for any Borel probability meausre μ on $\mathbb{R}^n$.

Haase [8] studies the dimension of product measures in terms of dim and Dim, here we discuss the case in terms of $\widehat{\delta}$ and Dim. In this section, we will restrict discussion to $\mathbb{R}^2$ in order to simplify notation, all our results have obvious analogs in higher dimensions.

Suppose that $E \subset \mathbb{R}^2$ and let A be a subset of the x-axis. For $a \in A$, denote $E_a = E \cap \{(x,y) : x = a\}$. Define $E_a^1(\varepsilon)$ to be the 1-dimensional closed ε-neighborhood of E_a on the direction of y-axis. For example, if $E_a = \{(x,y) : x = a,\ 1 \leq y \leq 2\}$, then $E_a^1(\varepsilon) = \{(x,y) : x = a,\ 1-\varepsilon \leq y \leq 2+\varepsilon\}$. Denote $a(\varepsilon)$ to be the 1-dimensional closed ε-neighborhood of a on x-axis, that is, $a(\varepsilon) = \{(x,y) : a-\varepsilon \leq x \leq a+\varepsilon,\ y = 0\}$.

Theorem 5.1. *Let E be a subset in $\mathbb{R}^2$ and let A be any subset of the x-axis. Suppose that if $x \in A$, $\mathcal{M}_*^t(E_x) > c$ for some constant c. Then $\mathcal{M}_*^{s+t}(E) \geq 2^{s+t-2} c \mathcal{M}_*^s(A)$.*

Proof. For any bounded sets $\{E_i\}$ with $E = \bigcup_i E_i$, we have $E_x = (\bigcup_i E_i)_x = \bigcup_i (E_i)_x$.

For $x \in A$, we have $\mathcal{M}_*^t(E_x) > c$, which means that

$$c < \mathcal{M}_*^t(E_x) \leq \sum_{i=1}^{\infty} M_*^t((E_i)_x) = \sum_{i=1}^{\infty} \liminf_{\varepsilon \downarrow 0} \left\{\varepsilon^{t-1} \mathcal{L}^1\left((E_i^1)_x\left(\frac{\varepsilon}{2}\right)\right)\right\}, \tag{5.3}$$

so we have

$$\inf_{x\in A}\sum_{i=1}^{\infty}\liminf_{\varepsilon\downarrow 0}\left\{\varepsilon^{t-1}\mathcal{L}^1\left((E_i^1)_x\left(\frac{\varepsilon}{2}\right)\right)\right\}>c, \tag{5.4}$$

then

$$\begin{aligned}
\sum_{i=1}^{\infty}M_*^{s+t}(E_i) &= \sum_{i=1}^{\infty}\liminf_{\varepsilon\downarrow 0}\{(2\varepsilon)^{s+t-2}\mathcal{L}^2(E_i(\varepsilon))\}\\
&\geq \sum_{i=1}^{\infty}\liminf_{\varepsilon\downarrow 0}\left\{(2\varepsilon)^{s+t-2}\mathcal{L}^2\left(\bigcup_{x\in A}((E_i)_x(\varepsilon))\right)\right\}\\
&\geq \sum_{i=1}^{\infty}\liminf_{\varepsilon\downarrow 0}\left\{(2\varepsilon)^{s+t-2}\mathcal{L}^2\left(\bigcup_{x\in A}\left((E_i^1)_x\left(\frac{\varepsilon}{2}\right)\times x\left(\frac{\varepsilon}{2}\right)\right)\right)\right\}\\
&\geq 2^{s+t-2}\inf_{x\in A}\sum_{i=1}^{\infty}\liminf_{\varepsilon\downarrow 0}\left\{\varepsilon^{s-1}\mathcal{L}^1\left(\bigcup_{x\in A}x\left(\frac{\varepsilon}{2}\right)\right)\varepsilon^{t-1}\mathcal{L}^1\left((E_i^1)_x\left(\frac{\varepsilon}{2}\right)\right)\right\}\\
&\geq 2^{s+t-2}\inf_{x\in A}\sum_{i=1}^{\infty}\liminf_{\delta\downarrow 0}\left\{\delta^{s-1}\mathcal{L}^1\left(A\left(\frac{\delta}{2}\right)\right)\right\}\liminf_{\varepsilon\downarrow 0}\left\{\varepsilon^{t-1}\mathcal{L}^1\left((E_i^1)_x\left(\frac{\varepsilon}{2}\right)\right)\right\}\\
&=2^{s+t-2}\liminf_{\delta\downarrow 0}\left\{\delta^{s-1}\mathcal{L}^1\left(A\left(\frac{\delta}{2}\right)\right)\right\}\inf_{x\in A}\sum_{i=1}^{\infty}\liminf_{\varepsilon\downarrow 0}\left\{\varepsilon^{t-1}\mathcal{L}^1\left((E_i^1)_x\left(\frac{\varepsilon}{2}\right)\right)\right\}.
\end{aligned} \tag{5.5}$$

The last line of the above inequality is bounded below by $2^{s+t-2}cM_*^s(A)$. Hence, we have

$$\mathcal{M}_*^{s+t}(E)\geq 2^{s+t-2}c\mathcal{M}_*^s(A) \tag{5.6}$$

since $M_*^s(A)\geq\mathcal{M}_*^s(A)$ and by the arbitrariness of $\{E_i\}$. □

Lemma 5.2. *For any subset E in $\mathbb{R}^2$, one has*

$$\mathcal{M}_*^{s+t}(E)\geq 2^{s+t-2}\int\mathcal{M}_*^t(E_x)\,d\mathcal{M}_*^s(x). \tag{5.7}$$

Proof. For any $\varepsilon>0$, there exists a sequence $0<c_1<\cdots<c_n<\cdots$ such that

$$\int\mathcal{M}_*^t(E_x)\,d\mathcal{M}_*^s-\varepsilon<\sum_n c_n\mathcal{M}_*^s(\{x:c_n<\mathcal{M}_*^t(E_x)\leq c_{n+1}\}). \tag{5.8}$$

Let $A_n=\{x:c_n<\mathcal{M}_*^t(E_x)\leq c_{n+1}\}$, $E_n=\bigcup\{E_x:x\in A_n\}$ for all n. By Theorem 5.1, we have

$$2^{s+t-2}\left(\int\mathcal{M}_*^t(E_x)\,d\mathcal{M}_*^s-\varepsilon\right)<\sum_n 2^{s+t-2}c_n\mathcal{M}_*^s(A_n)\leq\sum_n\mathcal{M}_*^{s+t}(E_n)=\mathcal{M}_*^{s+t}(E). \tag{5.9}$$

□

Theorem 5.3. *Let E be a subset in $\mathbb{R}^2$ and let A be any subset of the x-axis. Suppose that if $x\in A$, $\mathcal{M}_*^t(E_x)>c$ for some constant c. Then $\mathcal{P}^{s+t}(E)\geq(c/2)\mathcal{P}^s(A)$.*

Proof. For $x \in A$, we have $M_*^t(E_x) > c$, which means that

$$\liminf_{\varepsilon\downarrow 0}\{(2\varepsilon)^{t-1}\mathcal{L}^1(E_x(\varepsilon))\} > c, \tag{5.10}$$

so there exists $\delta_x > 0$ such that $(2\varepsilon)^{t-1}\mathcal{L}^1(E_x(\varepsilon)) > c$ when $0 < \varepsilon < \delta_x$.

Let $\delta > 0$ and $A_\delta = \{x \in A : (2\varepsilon)^{t-1}\mathcal{L}^1(E_x(\varepsilon)) > c,\ 0 < \varepsilon < \delta\}$, then $A_{\delta_1} \subset A_{\delta_2}$ as $\delta_1 > \delta_2$, which implies that $A_\delta \uparrow A$ as $\delta \downarrow 0$. Hence for $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $\delta \le \delta(\varepsilon)$,

$$\mathcal{P}^s(A_\delta) > \mathcal{P}^s(A) - \varepsilon \tag{5.11}$$

by the continuity of the measure $\mathcal{P}^s$. Let us first prove that

$$P^{s+t}(E) \ge \frac{c}{2}\mathcal{P}^s(A). \tag{5.12}$$

By the definitions of P^s and $\mathcal{P}^s$, we have

$$P_r^s(A_\delta) \ge P^s(A_\delta) \ge \mathcal{P}^s(A_\delta) \ge \mathcal{P}^s(A) - \varepsilon \tag{5.13}$$

for all $r > 0$ and $\delta \le \delta(\varepsilon)$, so $P_r^s(A_\delta) > \mathcal{P}^s(A) - \varepsilon$ holds for $r < \delta \le \delta(\varepsilon)$, thus there exists a family of disjoint closed intervals $\{I_i\}$ centered at A_δ and $|I_i| \le 2r$ for all i, say I_i has the center $x_i \in A_\delta \subset A$, such that $\sum_i |I_i|^s > \mathcal{P}^s(A) - \varepsilon$.

For each $x_i \in A_\delta, |I_i|/2 \le r < \delta$, so we have

$$|I_i|^{t-1}\mathcal{L}^1\left(E_{x_i}\left(\frac{|I_i|}{2}\right)\right) > c, \tag{5.14}$$

thus

$$N\left(E_{x_i}, \frac{|I_i|}{2}\right)a_1|I_i|\,|I_i|^{t-1} > |I_i|^{t-1}\mathcal{L}^1\left(E_{x_i}\left(\frac{|I_i|}{2}\right)\right) > c, \tag{5.15}$$

by Lemma 2.2 where $a_1 = 2$. Since the covering number and the packing number agree on the real line E_{x_i}, so we have

$$2P\left(E_{x_i}, \frac{|I_i|}{2}\right)|I_i|^t > c. \tag{5.16}$$

More precisely, there exist $P(E_{x_i}, |I_i|/2)$ disjoint closed intervals, centered at E_{x_i}, whose each length is $|I_i|$ such that $P(E_{x_i}, |I_i|/2)|I_i|^t > c/2$. Let $\{y_{ij}\}$ with $j = 1,2,\dots,P(E_{x_i}, |I_i|/2)$

be the centers of these intervals. Then all the balls centered at (x_i, y_{ij}), with radius $|I_i|/2 < r$, are disjoint which implies that they form a r-packing of E. Thus

$$\begin{aligned} P_r^{s+t}(E) &\geq \sum_{i=1}^{\infty} P\left(E_{x_i}, \frac{|I_i|}{2}\right) |I_i|^{s+t} \\ &= \sum_{i=1}^{\infty} \left(P\left(E_{x_i}, \frac{|I_i|}{2}\right) |I_i|^{t}\right) |I_i|^{s} \\ &> \frac{c}{2} \sum_{i=1}^{\infty} |I_i|^{s} \geq \frac{c}{2}\left(\mathcal{P}^s(A) - \varepsilon\right). \end{aligned} \tag{5.17}$$

It follows that

$$P^{s+t}(E) \geq \frac{c}{2}\mathcal{P}^s(A). \tag{5.18}$$

By Taylor and Tricot [4, Lemma 5.1], one has

$$\mathcal{P}^{s+t}(E) = \inf\left\{\lim_{n\to\infty} P^{s+t}(E_n) : E_n \uparrow E\right\}. \tag{5.19}$$

For any $E_n \uparrow E$, let

$$A_n = \{x \in A : \mathcal{M}_*^t\left(\left(E_n\right)_x\right) > c\}, \tag{5.20}$$

then by our intermediate result,

$$P^{s+t}(E_n) \geq \frac{c}{2}\mathcal{P}^s(A_n), \tag{5.21}$$

we have

$$\lim_{n\to\infty} P^{s+t}(E_n) \geq \frac{c}{2} \lim_{n\to\infty} \mathcal{P}^s(A_n). \tag{5.22}$$

Finally it suffices to verify that $\bigcup_n A_n = A$. First, $E_n \subset E_{n+1}$ implies that $A_n \subset A_{n+1}$. For any $x \in A$, we have $\mathcal{M}_*^t(E_x) > c$, since $\bigcup_n (E_n)_x = E_x$, and by the continuity of $\mathcal{M}_*^t$ there exists n_0 such that $\mathcal{M}_*^t(E_{n_0})_x > c$, which implies that $x \in A_{n_0}$, thus

$$\lim_{n\to\infty} P^{s+t}(E_n) \geq \frac{c}{2}\mathcal{P}^s(A) \tag{5.23}$$

by the continuity of $\mathcal{P}^s$, then

$$\mathcal{P}^{s+t}(E) \geq \frac{c}{2}\mathcal{P}^s(A) \tag{5.24}$$

by the arbitrariness of $\{E_n\}$. □

Remark 5.4. It is easy to see that [8, Lemma 5] is a corollary of Theorem 5.3 with only a different constant c since $\mathcal{M}_*^t(E) \geq 2^{-t}\mathcal{H}^t(E)$ by Theorem 4.4.

Lemma 5.5. *For any subset E in $\mathbb{R}^2$, one has*

$$\mathcal{P}^{s+t}(E) \geq \frac{1}{2}\int \mathcal{M}_*^t(E_x)\,d\mathcal{P}^s(x). \tag{5.25}$$

Proof. By Theorem 5.3, the proof is similar to that of Lemma 5.2. □

Now we are in a position to prove the following inequality.

Theorem 5.6. *For Borel probability measures μ, ν on $\mathbb{R}^2$, one has the following inequality:*

$$\hat{\delta}(\mu)+\hat{\delta}(\nu) \leq \hat{\delta}(\mu\times\nu) \leq \hat{\delta}(\mu)+\operatorname{Dim}(\nu) \leq \operatorname{Dim}(\mu\times\nu) \leq \operatorname{Dim}(\mu)+\operatorname{Dim}(\nu). \tag{5.26}$$

Proof. (i) By Lemma 5.2, the proof of the first inequality is similar to that of [8, Lemma 1].

(ii) The proof of the second inequality: for any $\varepsilon > 0$, choose Borel subsets E, F of $\mathbb{R}^n$ with

$$\begin{gathered}\hat{\delta}(E) < \hat{\delta}(\mu)+\frac{\varepsilon}{2}, \quad \mu(E) > 0,\\ \operatorname{Dim}(F) < \operatorname{Dim}(\nu)+\frac{\varepsilon}{2}, \quad \nu(F) > 0,\end{gathered} \tag{5.27}$$

then $\mu\times\nu(E\times F) > 0$, and we have

$$\hat{\delta}(\mu\times\nu) \leq \hat{\delta}(E\times F) \leq \hat{\delta}(E)+\operatorname{Dim}(F) < \hat{\delta}(\mu)+\operatorname{Dim}(\nu)+\varepsilon, \tag{5.28}$$

the second inequality above follows from Theorem 3.2, hence

$$\hat{\delta}(\mu\times\nu) \leq \hat{\delta}(\mu)+\operatorname{Dim}(\nu). \tag{5.29}$$

(iii) By Lemma 5.5, the proof of the third inequality is similar to that of [8, Lemma 7].

(iv) The last inequality is similar to that of [8, Lemma 3]. □

Remark 5.7. By a result of Tricot [5], we know that $\operatorname{Dim} = \hat{\Delta}$. Therefore, the conclusion of Theorem 5.6 can be rewritten as

$$\hat{\delta}(\mu)+\hat{\delta}(\nu) \leq \hat{\delta}(\mu\times\nu) \leq \hat{\delta}(\mu)+\hat{\Delta}(\nu) \leq \hat{\Delta}(\mu\times\nu) \leq \hat{\Delta}(\mu)+\hat{\Delta}(\nu). \tag{5.30}$$

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