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Research Article

On $\overline{\lambda}$-Statistically Convergent Double Sequences of Fuzzy Numbers

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We study the notion of $\overline{\lambda}$-statistically convergent for double sequence of fuzzy numbers and also get some inclusion relations.

1. Introduction

Nanda [1] studied sequence of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Nuray [2] proved the inclusion relations between the set of statistically convergent and lacunary statistically convergent sequences of fuzzy numbers. Kwon and Shim [3] studied statistical convergence and lacunary statistical convergence of sequences of fuzzy numbers, and they showed that Nuray's conditions are sufficient as well as necessary. Savaş [4] introduced and discussed double convergent sequence of fuzzy numbers and showed that the set of all double convergent sequences of fuzzy numbers is complete. In [5], Savaş generalized the statistical convergence by using de la Vallee-Poussin mean. Quite recently, Savaş and Mursaleen [6] introduced of statistically convergent and statistically Cauchy for double sequence of fuzzy numbers.

In this paper, we continue to study the concepts of strongly double $[V,\overline{\lambda}]$-summable and double $S_{\overline{\lambda}}$-convergent for double sequence of fuzzy numbers.

2. Preliminaries

Before continuing with the discussion, we pause to establish some notation. Let $C(R^n) = \{A \subset R^n : \text{A compact and convex}\}$. The spaces $C(R^n)$ have a linear structure induced by the operations

$$\begin{gathered} A + B = \{a + b,\ a \in A,\ b \in B\}, \\ \lambda A = \{\lambda a,\ \lambda \in A\} \end{gathered} \tag{2.1}$$

for $A,\ B \in C(R^n)$, and $\lambda \in R$. The Hausdorff distance between A and B of $C(R^n)$ is defined as

$$\delta_\infty(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}. \tag{2.2}$$

It is well known that $(C(R^n), \delta_\infty)$ is a complete (not separable) metric space.

A fuzzy number is a function X from R^n to $[0,1]$ satisfying

(1) X is normal, that is, there exists an $x_0 \in R^n$ such that $X(x_0) = 1$;

(2) X is fuzzy convex, that is, for any $x, y \in R^n$ and $0 \le \lambda \le 1$,

$$X(\lambda x + (1-\lambda)y) \ge \min\{X(x), X(y)\}; \tag{2.3}$$

(3) X is upper semicontinuous;

(4) the closure of $\{x \in R^n : X(x) > 0\}$, denoted by X^0, is compact.

These properties imply that for each $0 < \alpha \le 1$, the α-level set

$$X^\alpha = \{x \in R^n : X(x) \ge \alpha\} \tag{2.4}$$

is a nonempty compact convex, subset of R^n, as is the support X^0. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces addition $X + Y$ and scalar multiplication λX, $\lambda \in R$, in terms of α-level sets by

$$\begin{gathered}[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha, \\ [\lambda X]^\alpha = \lambda [X]^\alpha\end{gathered} \tag{2.5}$$

for each $0 \le \alpha \le 1$.

Define for each $1 \le q < \infty$,

$$d_q(X,Y) = \left\{\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha\right\}^{1/q} \tag{2.6}$$

and $d_\infty = \sup_{0\le\alpha\le1}\delta_\infty(X^\alpha, Y^\alpha)$. Clearly, $d_\infty(X,Y) = \lim_{q\to\infty} d_q(X,Y)$ with $d_q \le d_r$ if $q \le r$. Moreover, d_q is a complete, separable, and locally compact metric space [7].

Throughout the paper, d will denote d^q with $1 \le q \le \infty$.

We will need the following definitions.

Definition 2.1. A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be convergent in the Pringsheim's sense or P-convergent to a fuzzy number X_0 if for every $\varepsilon > 0$, there exists $N \in \mathcal{N}$ such that

$$d(X_{kl}, X_0) < \epsilon \quad \text{for } k, l > N, \tag{2.7}$$

and we denote $P - \lim X = X_0$. The number X_0 is called the Pringsheim limit of X_{kl}.

More exactly, we say that a double sequence (X_{kl}) converges to a finite number X_0 if X_{kl} tend to X_0 as both k and l tends to ∞ independently of one another.

Let $c^2(F)$ denote the set of all double convergent sequences of fuzzy numbers.

Definition 2.2. A double sequence $X = (X_{kl})$ of fuzzy numbers is bounded if there exists a positive number M such that $d(X_{kl}, X_0) < M$ for all k and l,

$$\|x\|_{(\infty,2)} = \sup_{k,l} d\left(X_{kl}, X_0\right) < \infty. \tag{2.8}$$

We will denote the set of all bounded double sequences by $l_\infty^2(F)$.

Let $K \subseteq \mathcal{N} \times \mathcal{N}$ be a two-dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i,j) in K such that $i \le n$ and $j \le m$. Then the lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K). \tag{2.9}$$

In the case when the sequence $(K_{m,n}/mn)_{m,n=1,1}^{\infty,\infty}$ has a limit, then we say that K has a natural density and is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K). \tag{2.10}$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \le P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0 \tag{2.11}$$

(i.e., the set K has double natural density zero).

Definition 2.3. A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be statistically convergent to X_0 provided that for each $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{nm} \left| \left\{ (j,k);\ j \le m,\ k \le n : d\left(X_{kl}, X_0\right) \ge \epsilon \right\} \right| = 0. \tag{2.12}$$

In this case, we write $\mathrm{st}_2 - \lim_{k,l} X_{k,l} = X_0$ and we denote the set of all double statistically convergent sequences of fuzzy numbers by $\mathrm{st}^2(F)$.

Definition 2.4. $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ could be two nondecreasing sequences of positive real numbers such that each tends to ∞ and

$$\begin{aligned} \lambda_{n+1} &\le \lambda_n + 1, \qquad \lambda_1 = 1, \\ \mu_{m+1} &\le \mu_m + 1, \qquad \mu_1 = 1. \end{aligned} \tag{2.13}$$

A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be $\overline{\lambda}$-summable if there is fuzzy number X_0 such that

$$P - \lim_{nm} \frac{1}{\overline{\lambda}_{nm}} \sum_{k \in I_n} \sum_{l \in I_m} d\left(X_{kl}, X_0\right) = 0, \tag{2.14}$$

where $I_n = [n - \lambda_n + 1, n]$, $I_m = [m - \mu_m + 1, m]$, and $\overline{\lambda}_{nm} = \lambda_n \mu_m$.

In this case, we say that X is strongly double $\overline{\lambda}$-summable to X_0 and we denote the set of all strongly double $\overline{\lambda}$-summable sequences by $[V,\overline{\lambda}](F)$. If $\overline{\lambda}_{nm} = nm$, then strongly double $\overline{\lambda}$-summable reduces to $[C,1,1](F)$, the space of strongly double Cesàro summable sequences defined as follows:

$$P - \lim_{nm} \frac{1}{nm} \sum_{k,l=1,1}^{mn} d\left(X_{kl}, X_0\right) = 0. \tag{2.15}$$

Definition 2.5. A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be double $\overline{\lambda}$-statistically convergent or $S_{\overline{\lambda}}$-convergent to X_0 if for every $\epsilon > 0$,

$$P - \lim_{n,m} \frac{1}{\overline{\lambda}_{nm}} \left|\left\{k \in I_n, l \in I_m : d\left(X_{kl}, X_0\right) \geq \epsilon\right\}\right| = 0. \tag{2.16}$$

In this case, we write $S_{\overline{\lambda}} - \lim X = X_0$ or $X_{kl} \xrightarrow{P} X_0(S_{\overline{\lambda}})$ and we denote the set of all double $S_{\overline{\lambda}}$-statistically convergent sequences of fuzzy numbers by $(S_{\overline{\lambda}})(F)$.

If $\overline{\lambda}_{nm} = nm$, for all n, m, then the set $S_{\overline{\lambda}}(F)$ of $S_{\overline{\lambda}}$-convergent sequences reduces to the space $\mathrm{st}^2(F)$.

We need the following proposition in future. A metric d on $L(\mathbb{R})$ is said to be a translation invariant if $d(X+Z, Y+Z) = d(X,Y)$ for $X, Y, Z \in L(\mathbb{R})$.

Proposition 2.6. *If d is a translation invariant metric on $L(\mathbb{R})$, then*

$$d(X+Y, 0) \leq d(X, 0) + d(Y, 0). \tag{2.17}$$

Proof is clear so we omitted it.

In the next theorem, we give some connections between strongly double $\overline{\lambda}$-summable and double $\overline{\lambda}$-statistical convergences.

3. Main results

Theorem 3.1. *A double sequence $X = (X_{kl})$ of fuzzy numbers is strongly double $\overline{\lambda}$-summable X_0, then it is double $\overline{\lambda}$-statistically convergent to X_0.*

Proof. Let $\epsilon > 0$ and since

$$\sum_{k \in I_n, l \in I_m} d\left(X_{kl}, X_0\right) \geq \sum_{k \in I_n, l \in I_m, d\left(X_{kl}, X_0\right) \geq \epsilon} d\left(X_{kl}, X_0\right) \geq \epsilon \left|\left\{k \in I_n, l \in I_m : d\left(X_{kl}, X_0\right) \geq \epsilon\right\}\right|. \tag{3.1}$$

This implies that if a sequence $X = (X_{kl})$ is strongly double $\overline{\lambda}$-summable X_0, then X is double $\overline{\lambda}$-statistically convergent to X_0.

This completes the proof. □

We have the following theorem.

Theorem 3.2. *If a bounded (X_{kl}) is double $\overline{\lambda}$-statistically convergent to X_0, then it is strongly double $\overline{\lambda}$-summable X_0.*

Proof. Suppose that (X_{kl}) is bounded and double $\overline{\lambda}$-statistically convergent to X_0. Since X is bounded we write $d(X_{kl}, X_0) \leq M$ for all k, l. Also for given $\epsilon > 0$ and n and m large we obtain

$$\frac{1}{\overline{\lambda}_{nm}} \sum_{k\in I_n, l\in I_m} d(X_{kl}, X_0) = \frac{1}{\overline{\lambda}_{nm}} \sum_{k\in I_n, l\in I_m, d(X_{kl}, X_0)\geq\epsilon} d(X_{kl}, X_0) + \frac{1}{\overline{\lambda}_{nm}} \sum_{k\in I_n, l\in I_m, d(X_{kl}, X_0)<\epsilon} d(X_{kl}, X_0)$$
$$\leq \frac{M}{\overline{\lambda}_{nm}} \left|\{k \in I_n,\ l \in I_m : d(X_{kl}, X_0) \geq \epsilon\}\right| + \epsilon, \tag{3.2}$$

which implies that X is strongly double $\overline{\lambda}$-summable X_0.

This completes the proof. □

Theorem 3.3. *If a sequence $X = (X_{kl})$ of fuzzy numbers is double statistically convergent to X_0, then it is double $\overline{\lambda}$-statistically convergent to X_0 if and only if*

$$P - \lim_{nm} \inf \frac{\overline{\lambda}_{nm}}{nm} > 0. \tag{3.3}$$

Proof. For given $\varepsilon > 0$, we have

$$\{k \leq n,\ l \leq m : d(X_{kl}, X_0) \geq \epsilon\} \supset \{k \in I_n,\ l \in I_m : d(X_{kl}, X_0) \geq \epsilon\}. \tag{3.4}$$

Therefore,

$$\frac{1}{nm}\left|\{k \leq n,\ l \leq m : d(X_{kl}, X_0) \geq \epsilon\}\right| \geq \frac{1}{nm}\left|\{k \in I_n,\ l \in I_m : d((X_{kl}, X_0) \geq \epsilon\}\right|$$
$$\geq \frac{\overline{\lambda}_{nm}}{nm} \frac{1}{\overline{\lambda}_{nm}} \left|\{k \in I_n,\ l \in I_m : d((X_{kl}, X_0) \geq \epsilon\}\right|. \tag{3.5}$$

Taking the limit as $n, m \to \infty$ and using hypothesis, we get X is double $\overline{\lambda}$-statistically convergent to X_0.

Conversely, suppose that $X \in \mathrm{st}_2(F)$ and since $\overline{\lambda}_{nm} = \lambda_n \mu_m$, either $P - \lim_n \inf \lambda_n/n = 0$ or $P - \lim_m \inf (\mu_m/m) = 0$ or both are zero. Then we can choose subsequences $(n(p))_{p=1}^{\infty}$ and $(m(q))_{q=1}^{\infty}$ such that $\lambda_{n_{(p)}}/n(p) < 1/p$ and $\mu_{m_{(q)}}/m(q) < 1/q$. Define a sequence $X = (X_{kl})$ by

$$X_{kl} = \begin{cases} 1 & \text{if } k \in I_{n_{(p)}},\ l \in I_{m(q)}\ (p,\ q = 1, 2, \ldots), \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Then $X \in [C, 1, 1](F)$ and hence, by [6, Theorem 6(a)], $X \in \mathrm{st}^2(F)$. But on the other hand, $X \notin [V, \overline{\lambda}](F)$ and from Theorem 3.1, $X \notin (S_{\overline{\lambda}})(F)$; a contradiction and hence (3.3) must hold. □

Finally, we conclude this paper by stating a definition which generalizes Definition 2.4.

Definition 3.4. Let $X = (X_{kl})$ be a double sequence of fuzzy numbers and let p be positive real numbers. The sequence X is said to be strongly double $\overline{\lambda}_p$-summable if there is fuzzy number X_0 such that

$$P - \lim_{nm} \frac{1}{\overline{\lambda}_{nm}} \sum_{k \in I_n} \sum_{l \in I_m} d(X_{kl}, X_0)^p = 0. \tag{3.7}$$

In this case, we say that X is strongly double $\overline{\lambda}_p$-summable to X_0. If $\overline{\lambda}_{nm} = nm$, then strongly double $\overline{\lambda}_p$-summable reduces to strongly double p-Cesàro summable to X_0.

Theorem 3.5. (1) *Let $p \in (0, \infty)$. If a double sequence $X = (X_{kl})$ of fuzzy numbers is strongly double $\overline{\lambda}_p$-summable X_0, then it is double $\overline{\lambda}$-statistically convergent to X_0.*

(2) *Let $p \in (0, \infty)$. If a bounded (X_{kl}) is double $\overline{\lambda}$-statistically convergent to X_0, then it is strongly double $\overline{\lambda}_p$-summable X_0.*

Proof. The proof of theorem is similar to that of Theorems 3.1 and 3.2 so we omitted it. □

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