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Research Article
On the Cesáro Summability of Double Series

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In a recent paper by Savaş and Şevli (2007), it was shown that each Cesáro matrix of order α, for $\alpha > -1$, is absolutely kth power conservative for $k \geq 1$. In this paper we extend this result to double Cesáro matrices.

The concept of absolute summability of order $k \geq 1$ was defined by Flett [1] as follows. Let $\sum a_k$ be a series with partial sums (s_n), A an infinite matrix. Then $\sum a_k$ is said to be absolutely summable A of order $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} \left| T_{n-1} - T_n \right|^k < \infty, \tag{1}$$

where

$$T_n := \sum_{k=0}^{\infty} a_{nk} s_k. \tag{2}$$

Denote by $\mathcal{A}_k$ the sequence space defined by

$$\mathcal{A}_k = \left\{ (s_n) : \sum_{n=1}^{\infty} n^{k-1} \left| a_n \right|^k < \infty;\ a_n = s_n - s_{n-1} \right\} \tag{3}$$

for $k \geq 1$. A matrix T is said to be a bounded linear operator on $\mathcal{A}_k$, written $T \in B(\mathcal{A}_k)$, if $T : \mathcal{A}_k \to \mathcal{A}_k$. In 1970, Das [2] defined such a matrix to be absolutely kth power conservative for $k \geq 1$. In that paper, he proved that every conservative Hausdorff matrix $H \in B(\mathcal{A}_k)$ for $k \geq 1$. In a recent paper [3], the first two authors proved every Cesáro matrix of order α, for $\alpha > -1$, $(C, \alpha) \in B(\mathcal{A}_k)$ for $k \geq 1$. Since the Cesáro matrices of order α for $-1 < \alpha < 0$ are not conservative, their result shows that being conservative is not a necessary condition for being absolutely kth power conservative.

In this paper, we extend the result of [3] to double summability, thereby demonstrating that the property of being conservative is again not necessary for doubly infinite matrices to be absolutely kth power conservative.

Let $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}$ be an infinite double series with real or complex numbers, with partial sums

$$s_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}. \tag{4}$$

For any double sequence (x_{mn}), we will define

$$\Delta_{11} x_{mn} = x_{mn} - x_{m+1,n} - x_{m,n+1} + x_{m+1,n+1}. \tag{5}$$

The series $\sum\sum a_{mn}$ is said to be summable $|C, \alpha, \beta|_k$, $k \geq 1$, α, $\beta > -1$, if (see [4])

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^{k-1} \left|\Delta_{11} \sigma_{m-1,n-1}^{\alpha\beta}\right|^k < \infty, \tag{6}$$

where $\sigma_{mn}^{\alpha\beta}$ denotes the mn-term of the (C, α, β) transform of a sequence (s_{mn}), that is,

$$\sigma_{mn}^{\alpha\beta} = \frac{1}{E_m^{\alpha} E_n^{\beta}} \sum_{i=0}^{m}\sum_{j=0}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} s_{ij}. \tag{7}$$

Define

$$\mathcal{A}_k^2 := \left\{ (s_{mn})_{m,n=0}^{\infty} : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^{k-1} \left|a_{mn}\right|^k < \infty;\ a_{mn} = \Delta_{11} s_{m-1,n-1} \right\} \tag{8}$$

for $k \geq 1$.

A four-dimensional matrix $T = (t_{mnij} : m, n, i, j = 0, 1, \ldots)$ is said to be absolutely kth power conservative, for $k \geq 1$, if $T \in B(\mathcal{A}_k^2)$; that is, if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^{k-1} \left|\Delta_{11} s_{m-1,n-1}\right|^k < \infty \tag{9}$$

implies that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (mn)^{k-1} \left|\Delta_{11} t_{m-1,n-1}\right|^k < \infty, \tag{10}$$

where

$$t_{mn} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t_{mnij} s_{ij} \quad (m, n = 0, 1, \ldots). \tag{11}$$

Theorem 1. $(C,\alpha,\beta) \in B(\mathscr{A}_k^2)$ *for each* $\alpha,\beta > -1$.

Proof. Let $\tau_{mn}^{\alpha\beta}$ denote the mn-term of the (C,α,β)-transform, in terms of (mna_{mn}); that is,

$$\tau_{mn}^{\alpha\beta} = \frac{1}{E_m^\alpha E_n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n} E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1} ij a_{ij}. \tag{12}$$

For $\alpha,\ \beta > -1$, since

$$\tau_{mn}^{\alpha\beta} = mn\left(\sigma_{mn}^{\alpha\beta} - \sigma_{m,n-1}^{\alpha\beta} - \sigma_{m-1,n}^{\alpha\beta} + \sigma_{m-1,n-1}^{\alpha\beta}\right), \tag{13}$$

to prove the theorem, it will be sufficient to show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^k < \infty. \tag{14}$$

Using Hölder's inequality, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^k &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}ij a_{ij}\right|^k \\ &\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mnE_m^\alpha E_n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}(ij)^k\left|a_{ij}\right|^k \times\left\{\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}\right\}^{k-1}.\end{aligned} \tag{15}$$

Since

$$\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1} = 1, \tag{16}$$

we obtain

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^k &\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mnE_m^\alpha E_n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}(ij)^k\left|a_{ij}\right|^k \\ &\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^k\left|a_{ij}\right|^k\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}}{mnE_m^\alpha E_n^\beta}.\end{aligned} \tag{17}$$

For $\alpha,\ \beta > -1$ and $m,n \ge 1$,

$$\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}}{mnE_m^\alpha E_n^\beta} = \sum_{m=i}^{\infty}\frac{E_{m-i}^{\alpha-1}}{mE_m^\alpha}\sum_{n=j}^{\infty}\frac{E_{n-j}^{\beta-1}}{nE_n^\beta} = \frac{1}{j}\sum_{m=i}^{\infty}\frac{E_{m-i}^{\alpha-1}}{mE_m^\alpha} = (ij)^{-1}. \tag{18}$$

Thus

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\left|\tau_{mn}^{\alpha\beta}\right|^k = O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^k\left|a_{ij}\right|^k\frac{1}{ij} = O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}\left|a_{ij}\right|^k = O(1) \tag{19}$$

since $(s_{mn}) \in \mathscr{A}_k^2$. □

Using the notation of [5],

$$\begin{aligned}
\theta_{mn}^{\alpha} &:= \frac{1}{E_m^{\alpha}} \sum_{i=0}^{m} E_{m-i}^{\alpha-1} s_{in} = (C,\alpha,0)(s_{mn}), \\
\theta_{mn}^{\beta} &:= \frac{1}{E_n^{\beta}} \sum_{j=0}^{n} E_{n-j}^{\beta-1} s_{mj} = (C,0,\beta)(s_{mn}), \\
\sigma_{mn} &:= \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} s_{ij} = (C,1,1)(s_{mn}).
\end{aligned} \tag{20}$$

Corollary 1. $(C,\alpha,0) \in B(\mathcal{A}_k^2)$ *for each* $\alpha > -1$.

Corollary 2. $(C,0,\beta) \in B(\mathcal{A}_k^2)$ *for each* $\alpha > -1$.

Corollary 3. $(C,1,1) \in B(\mathcal{A}_k^2)$.

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