Research Article

On Interpolation Functions of the Generalized Twisted \((h,q)\)-Euler Polynomials

Kyoung Ho Park

Department of Mathematics, Sogang University, Seoul 121-742, South Korea

Correspondence should be addressed to Kyoung Ho Park, sagamath@yahoo.co.kr

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The aim of this paper is to construct \(p\)-adic twisted two-variable Euler-\((h,q)\)-L-functions, which interpolate generalized twisted \((h,q)\)-Euler polynomials at negative integers. In this paper, we treat twisted \((h,q)\)-Euler numbers and polynomials associated with \(p\)-adic invariant integral on \(\mathbb{Z}_p\). We will construct two-variable twisted \((h,q)\)-Euler-zeta function and two-variable \((h,q)\)-L-function in Complex \(s\)-plane.

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1. Introduction

Tsumura and Young treated the interpolation functions of the Bernoulli and Euler polynomials in [1, 2]. Kim and Simsek studied on \(p\)-adic interpolation functions of these numbers and polynomials [3–48]. In [49], Carlitz originally constructed \(q\)-Bernoulli numbers and polynomials. Many authors studied these numbers and polynomials [4, 28, 38, 41, 50]. After that, twisted \((h,q)\)-Bernoulli and Euler numbers(polynomial) s were studied by several authors [1–32, 32–65]. In [62], Whashington constructed one-variable \(p\)-adic-L-function which interpolates generalized classical Bernoulli numbers at negative integers. Fox introduced the two-variable \(p\)-adi L-functions [53]. Young defined \(p\)-adic integral representation for the two-variable \(p\)-adic L-functions [64]. Furthermore, Kim constructed the two-variable \(p\)-adic \(q\)-L-function, which is interpolation function of the generalized \(q\)-Bernoulli polynomials [8]. This function is the \(q\)-extension of the two-variable \(p\)-adic L-function. Kim constructed \(q\)-extension of the generalized formula for two-variable of Diamond and Ferrero and Greenberg formula for two-variable \(p\)-adic L-function in the terms of the \(p\)-adic gamma and log-gamma functions [8]. Kim and Rim introduced twisted \(q\)-Euler numbers and polynomials associated with basic twisted \(q\)-\(\ell\)-functions [28]. Also, Jang et al. investigated the \(p\)-adic analogue twisted \(q\)-\(\ell\)-function, which interpolates generalized twisted
q-Euler numbers $E_{n,q},\chi$ attached to Dirichlet’s character $\chi$ [55]. Kim et al. have studied two-variable $p$-adic $L$-functions, which interpolate the generalized Bernoulli polynomials at negative integers. In this paper, we will construct two-variable $p$-adic twisted Euler $(h,q)$-$L$-functions. This functions interpolation functions of the generalized twisted $(h,q)$-Euler polynomials.

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z},\mathbb{Z}_p,\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of rational integers, the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ such that $|p|_p = p^{-\nu_p(p)} = p^{-1}$. If $s \in \mathbb{C}$, then $|q|_p < 1$. If $q \in \mathbb{C}_p$, we normally assume $|1 - q|_p < p^{-(1/(p-1))}$, so that $q^x = \exp(\log q)$ for $|x|_p \leq 1$. Throughout this paper we use the following notations (cf. [1–32, 32–48, 50, 51, 54–65]):

$$ [x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \quad [x]_q = \frac{1 - (-q)^x}{1 + q}. \quad (1.1) $$

Hence, $\lim_{q \to 1} [x]_q = x$, for any $x$ with $|x|_p \leq 1$ in the present $p$-adic case.

For $d$ a fixed positive integer with $(p,d) = 1$, set

$$ X = X_d = \lim_{N \to \infty} \frac{\mathbb{Z}}{dp^NZ}, \quad X_1 = \mathbb{Z}_p, $$

$$ X^* = \bigcup_{0 < a < dp_1, (a,p) = 1} (a + dp\mathbb{Z}_p), \quad (1.2) $$

$$ a + dp^NZ = \{ x \in X \mid x \equiv a \pmod{dp^N} \}, $$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. The distribution is defined by

$$ \mu_q(a + dp^NZ_p) = \frac{q^a}{[dp^N]_q}. \quad (1.3) $$

We say that $f$ is uniformly differential function at a point $a \in \mathbb{Z}_p$, and we write $f \in UD(\mathbb{Z}_p)$, if the difference quotients, $F_j(x,y) = (f(x) - f(y))/(x - y)$ have a limit $f'(a)$ as $(x,y) \to (a,a)$.

For $f \in UD(\mathbb{Z}_p)$, the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$ is defined as [4, 18]

$$ I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x. \quad (1.4) $$

The fermionic $p$-adic $q$-measures on $\mathbb{Z}_p$ is defined as (cf. [14–16, 18, 22, 28])

$$ \mu_{-q}(a + dp^NZ_p) = \frac{(-q)^a}{[dp^N]_{-q}}, \quad (1.5) $$
for \( f \in UD(\mathbb{Z}_p) \). For \( f \in UD(\mathbb{Z}_p) \), the fermionic \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \) is defined as

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \to \infty} \frac{1}{|pN|_q} \sum_{x=0}^{pN-1} f(x)(-q)^x,
\]

which has a sense as we see readily that the limit is convergent. For \( f \in UD(\mathbb{Z}_p, \mathbb{C}_p) \), we note that (cf. [14, 16, 18, 22, 28])

\[
\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x).
\]

From the fermionic invariant integral on \( \mathbb{Z}_p \), we derive the following integral equation (cf. [14, 35]):

\[
I_{-1}(f_1) + I_{-1}(f) = 2f(0),
\]

where \( f_1(x) = f(x + 1) \).

### 2. Twisted \((h, q)\)-Euler Numbers and Polynomials

In this section, we will treat some properties of twisted \((h, q)\)-Euler numbers and polynomials associated with \( p \)-adic invariant integral on \( \mathbb{Z}_p \). From now on, we take \( h \in \mathbb{Z} \) and \( q \in \mathbb{C}_p \) with \( |q - 1|_p < p^{-(1/(p-1))} \). Let \( \mathbb{C}_{pn} \) be the space of primitive \( p^n \)th root of unity,

\[
\mathbb{C}_{pn} = \{ w \in \mathbb{C}_p : w^{p^n} = 1 \}.
\]

Then, we denote

\[
T_p = \lim_{n \to \infty} \mathbb{C}_{pn} = \bigcup_{n \geq 0} \mathbb{C}_{pn}.
\]

Hence \( T_p \) is a \( p \)-adic locally constant space. For \( \xi \in T_p \), we denote by \( \phi_\xi : \mathbb{Z}_p \to \mathbb{C}_p \) defined by \( \phi_\xi(x) = \xi^x \), the locally constant function. If we take \( f(x) = \xi^x e^{\xi x} \), then we have (cf. [35])

\[
E_{n,\xi} = \int_{\mathbb{Z}_p} x^n \xi^x d\mu_1(x).
\]

By induction in (1.8), Kim constructed the following useful identity (cf. [14, 28]):

\[
I_{-1}(f_n) + (-1)^n I_{-1}(f) = 2 \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} f(\ell),
\]
where \( n \in \mathbb{N} \), \( f_n = f(x + n) \). From (2.4), if \( n \) is odd, then we have

\[
L_1(f_n) + L_1(f) = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell} f(\ell). \tag{2.5}
\]

If we replace \( n \) by \( d \) (\( = \) odd) into (2.5), we obtain

\[
L_1(f_d) + L_1(f) = 2 \sum_{\ell=0}^{d-1} (-1)^{\ell} f(\ell). \tag{2.6}
\]

Let \( \xi \in T_p \). Let \( \chi \) be a Dirichlet’s character of conductor \( d \), which \( d \) is any multiple of \( p \) with \( p \equiv 1 \) (mod 2). By substituting \( f(x) = \chi(x)\xi^x e^{xt} \) into (2.6), we have

\[
L_1(\chi(x)\xi^x e^{xt}) = \sum_{n=1}^{\infty} E_{n,\xi,\chi} \frac{t^n}{n!}. \tag{2.7}
\]

Remark 2.1. In complex case, the generating function of the Euler numbers \( E_{n,\xi,\chi} \) is given by (cf. [28])

\[
2 \sum_{\ell=0}^{d-1} (-1)^{\ell} \chi(\ell)\xi^\ell e^{\ell t} = \sum_{n=0}^{\infty} E_{n,\xi,\chi} \frac{t^n}{n!}, \quad |t| < \frac{\pi}{d}. \tag{2.8}
\]

By using Taylor series of \( e^{xt} \), then we can define the generalized twisted Euler numbers \( E_{n,\xi,\chi} \) attached to \( \chi \) as follows (cf. [55]):

\[
E_{n,\xi,\chi} = \int_{\chi} \xi^n x^n \chi(x) d\mu_{-1}(x). \tag{2.9}
\]

In [8], \((h,q)\)-Euler numbers were defined by

\[
E_{n,q}^{(h,1)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)r} [x + y]_q^n d\mu_{q^{-q}}(y), \tag{2.10}
\]

where \( h \in \mathbb{Z} \) and \( x \in \mathbb{Z}_p \). In particular, if we take \( x = 0 \), then \( E_{n,q}^{(h,1)}(0) = E_{n,q}^{(h,1)} \). These numbers are called \((h,q)\)-Euler numbers.

By using iterative method of \( p \)-adic invariant integral on \( \mathbb{Z}_p \) in the sense of fermionic, we define twisted \((h,q)\)-Euler numbers as follows (cf. [55]):

\[
E_{n,q}^{(h,1)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)q} \phi(y) [x + y]_q^n d\mu_{q^{-q}}(y). \tag{2.11}
\]
For \( h \in \mathbb{Z} \) and \( n \in \mathbb{N} \), we have that (cf. [55])

\[
E^{(h,1)}_{n,q,t}(x) = \frac{1+q}{(1-q)^n} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (-1)^i q^i \frac{1}{1+q^{h+i}},
\]
(2.12)

\[
E^{(h,1)}_{n,q,t}(x) = \frac{1+q}{1+q^d} \sum_{a=0}^{d-1} (-1)^a q^{ha} \xi^a E^{(h,1)}_{n,q,t} \left( \frac{x+a}{d} \right) [d]^n\xi,
\]
(2.13)

where \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod 2 \).

Let \( F^{(h,1)}_{q,t} (t, x) \) be the generating function of \( E^{(h,1)}_{n,q,t}(x) \) in complex plane as follows (cf. [55]):

\[
F^{(h,1)}_{q,t}(t, x) = (1 + q) \sum_{n=0}^{\infty} (-1)^n q^{-n} t^n e^{[n+x]_q} = \sum_{n=0}^{\infty} E^{(h,1)}_{n,q,t}(x) \frac{t^n}{n!}.
\]
(2.14)

Let \( \chi \) be the Dirichlet’s character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod 2 \). Then the generalized twisted \((h, q)\)-Euler polynomials attached to \( \chi \) is given by as follows:

For \( n \in \mathbb{Z}_q = \mathbb{N} \cup \{0\} \),

\[
E^{(h,1)}_{n,q,t,\chi}(x) = \int_{\chi} \chi(y) q^{(-1)^n y} [x+y]_q^n d_{\mu,q}(y),
\]
(2.15)

where \( h \in \mathbb{Z}, d \) is any multiple of \( p \) with \( p \equiv 1 \pmod 2 \) and \( x \in \mathbb{C}_p \).

Then the distribution relation of the generalized twisted \((h, q)\)-Euler polynomials is given by as follows (cf. [14]):

\[
E^{(h,1)}_{n,q,t,\chi}(x) = \frac{1+q}{1+q^d} \sum_{a=1}^{d} \chi(a)(-1)^a q^{ha} \xi^a E^{(h,1)}_{n,q,t} \left( \frac{x+a}{d} \right) [d]^n\xi.
\]
(2.16)

### 3. Two-Variable Twisted \((h, q)\)-Euler-Zeta Function and \((h, q)\)-L-Function

In this section, we will construct two-variable twisted \((h, q)\)-Euler-zeta function and two-variable \((h, q)\)-L-function in Complex s-plane. We assume \( q \in \mathbb{C} \) with \(|q| < 1\).

Firstly, we consider twisted \( q \)-Euler numbers and polynomials in \( \mathbb{C} \) as follows (cf. [55]):

\[
F^{(h,1)}_{q,t}(t, x) = (1 + q) \sum_{n=0}^{\infty} (-1)^n q^{-n} t^n e^{[n+x]_q} = \sum_{n=0}^{\infty} E^{(h,1)}_{n,q,t}(x) \frac{t^n}{n!},
\]
(3.1)
where \( q, x \in \mathbb{C}, r \in \mathbb{Z}^* = \mathbb{N} \cup \{0\} \) and \( \zeta \) is rth root of unity. In particular, if we take \( x = 0 \), then we have \( E_{n,q,\zeta}^{(h,1)}(0) = E_{n,q,\zeta}^{(h,1)} \). These numbers are called twisted Euler numbers. By using derivative operator, we have \( \left( \frac{d^k}{dt^k} \right) F_{q,\zeta}(t, x) \bigg|_{t=0} = E_{n,q,\zeta}^{(h,1)}(x) \).

From (3.1), we can define Hurwitz-type twisted \((h, q)\)-Euler-zeta function as follows (cf. [55]):

\[
\xi_{E,q,\zeta}(s, x) = (1 + q) \sum_{k=0}^{\infty} \frac{(-1)^k q^k \xi^k}{[x + k]^q},
\]

where \( q \in \mathbb{C}, |q| < 1, s \in \mathbb{C}, h \in \mathbb{Z} \) and \( x \in \mathbb{R}, 0 < x \leq 1 \). Note that if \( x = 1 \) in (3.2), then we see that the twisted \((h, q)\)-Euler-zeta function is defined by (cf. [28, 55])

\[
\xi_{E,q,\zeta}(s) = (1 + q) \sum_{k=1}^{\infty} \frac{(-1)^k q^k \xi^k}{[k]^q}, \quad s \in \mathbb{C}, \text{Re}(s) > 1.
\]

For \( n \in \mathbb{N} \), we know (cf. [28])

\[
\xi_{E,q,\zeta}^{(h,1)}(-n, x) = E_{\eta,n,q,\zeta}^{(h,1)}(x).
\]

From now on, we will define the two-variable \((h, q)\)-functions \( L_{E,q,\zeta}^{(h,1)}(s, x : \chi) \) which interpolates the generalized \((h, q)\)-Euler polynomials.

**Definition 3.1.** Let \( \chi \) be the Dirichlet’s character with conductor \( d \) with \( d \equiv 1 (\text{mod} \ 2) \). For \( s \in \mathbb{C}, h \in \mathbb{Z} \) and \( x \in \mathbb{R}, 0 < x \leq 1 \), we define

\[
L_{E,q,\zeta}^{(h,1)}(s, x : \chi) = (1 + q) \sum_{n=0}^{\infty} \frac{\chi(n)(-1)^n q^n \xi^n}{[n + x]^q}.
\]

By substituting \( n = a + jd, d \equiv 1 (\text{mod} \ 2), 1 \leq a \leq d \) and \( n = 0, 1, 2, \ldots \) into (3.5), then using (3.2), we have

\[
L_{E,q,\zeta}^{(h,1)}(s, x : \chi)(1 + q) \sum_{a=1}^{d} \sum_{j=0}^{\infty} \frac{\chi(a)(a + jd)(-1)^{a+jd} q^{(a+jd)\xi} [a + jd + x]^q}{[a + jd + x]^q} = (1 + q) \sum_{a=1}^{d} \sum_{j=0}^{\infty} \frac{\chi(a)(a + jd)(-1)^{a+jd} q^{(a+jd)\xi} [a + jd + x]^q}{[a + jd + x]^q},
\]

Thus, we see the function \( L_{E,q,\zeta}^{(h,1)}(s, x : \chi) \) which interpolates the generalized \((h, q)\)-Euler polynomials as follows.
Theorem 3.2. For $s \in \mathbb{C}$, $h \in \mathbb{Z}$, let $\chi$ be the Dirichlet’s character with conductor $d$ with $d \equiv 1 \pmod{2}$. Then one has

$$L_{E,q,h}^{(s)}(s, x : \chi) = \frac{1 + q}{1 + q^d} \sum_{a=1}^{d} \chi(a) (-1)^{a} q^{rac{ha}{d}} \mathcal{B}_{E,q,h}^{(s)} \left( s, \frac{a + x}{d} \right) [d]_{q}^{-s}.$$  \hspace{1cm} (3.7)

By substituting $s = -n$ with $n > 0$, into (3.7), we obtain

$$L_{E,q,h}^{(s)}(-n, x : \chi) = \frac{1 + q}{1 + q^d} \sum_{a=1}^{d} \chi(a) (-1)^{a} q^{rac{ha}{d}} \mathcal{B}_{E,q,h}^{(s)} \left( -n, \frac{a + x}{d} \right) [d]_{q}^{n}$$

$$= \frac{1 + q}{1 + q^d} \sum_{a=1}^{d} \chi(a) (-1)^{a} q^{rac{ha}{d}} \mathcal{B}_{E,q,h}^{(s)} \left( \frac{a + x}{d} \right) [d]_{q}^{n}$$

$$= E_{n,q,h}^{(s)} (x),$$

where $d \equiv 1 \pmod{2}$, $d \in \mathbb{N}$.

Thus, we have the following theorem.

Theorem 3.3. For $n \in \mathbb{N}$, let $\chi$ be the Dirichlet’s character with conductor $d$ with $d \equiv 1 \pmod{2}$. Then one has

$$L_{E,q,h}^{(s)}(-n, x : \chi) = E_{n,q,h}^{(s)} (x).$$  \hspace{1cm} (3.9)

Remark 3.4. If we take $x = 1$ in (3.5), then we have (cf. [28, 55])

$$L_{E,q,h}^{(s)}(s, \chi) = (1 + q) \sum_{n=1}^{\infty} \chi(n) (-1)^{n} q^{rac{hn}{d}} [n]_{q}^{s}, \quad \text{for } s \in \mathbb{C}. \hspace{1cm} (3.10)$$

From (3.9) and (3.10), we have the following corollary.

Corollary 3.5. Let $\chi$ be the Dirichlet’s character with conductor $d$ with $d \equiv 1 \pmod{2}$. Then one has

$$E_{n,q,h}^{(s)} (x) = \frac{1 + q}{1 + q^d} \sum_{a=1}^{d} \chi(a) (-1)^{a} q^{rac{ha}{d}} \mathcal{B}_{n,q,h}^{(s)} \left( \frac{a + x}{d} \right) [d]_{q}^{n}. \hspace{1cm} (3.11)$$

Secondly, we will define two-variable twisted Euler $(h, q)$-$L$-function as follows.

Definition 3.6. Let $\chi$ be the Dirichlet’s character with conductor $d$ with $d \equiv 1 \pmod{2}$, $d \in \mathbb{N}$. For $s \in \mathbb{C}$, $h \in \mathbb{Z}$, $x \in \mathbb{R}$, $0 < x \leq 1$ and $\xi'$ with $\xi' \neq 1$, we define

$$L_{E,q,h}^{(s)}(s, x : \chi) = (1 + q) \sum_{k=0}^{\infty} \chi(k) (-1)^{k} q^{rac{hk}{d}} [k + x]_{q}^{s}. \hspace{1cm} (3.12)$$
We consider the well-known identity (cf. [44, 65])

$$\frac{1}{(1-x)^s} = \sum_{j=0}^{\infty} \binom{s+j-1}{j} x^j.$$  \hfill (3.13)

By using (3.12), we define two-variable twisted Euler \((h,q)\)-\(L\)-function as follows:

$$L_{E,q,h}^{(h,1)}(s, x : \chi) = (1+q)(1-q)^s \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{s+j-1}{j} \chi(k)(-1)^k q^k h_{q,k+j+kx}. \hfill (3.14)$$

We will investigate the relations between \(L_{E,q,h}^{(h,1)}(s, x : \chi)\) and \(L_{E,q,h}^{(h,1)}(s, \chi)\) as follows.

Substituting \(k = a+jd, a = 1, 2, \ldots, d\) with \(d \equiv 1 \pmod{2}, j = 0, 1, 2, \ldots, \) into (3.12), we have

$$L_{E,q,h}^{(h,1)}(s, x : \chi) = (1+q) \sum_{a=1}^{d} \sum_{j=0}^{\infty} \chi(a+jd)(-1)^{a+jd} q^{h(a+jd)} \frac{\sum_{j=0}^{\infty} \chi(j) q^j}{[a+jd+x]_q^s}, \hfill (3.15)$$

Thus we obtain the following theorem.

**Theorem 3.7.** For \(s \in \mathbb{C}\) with \(h \in \mathbb{Z}\), let \(\chi\) be the Dirichlet character with conductor \(d\) with \(d \equiv 1 \pmod{2}\) and \(x \in \mathbb{R}, 0 < x \leq 1, \xi^r = 1\) with \(\xi \neq 1\). Then one has

$$L_{E,q,h}^{(h,1)}(s, x : \chi) = \frac{1+q}{1+qd} \sum_{a=1}^{d} \chi(a)(-1)^a q^{h(a)} \frac{\varepsilon_{E,q,h}^{(h,1)}(s, a+x/d) [d]_q^{-s}}{d}. \hfill (3.16)$$

By substituting \(s = -n\) with \(n \in \mathbb{N}\) into (3.16) and using (3.4), we can obtain

$$L_{E,q,h}^{(h,1)}(-n, x : \chi) = \frac{1+q}{1+qd} \sum_{a=1}^{d} \chi(a)(-1)^a q^{h(a)} \frac{\varepsilon_{E,q,h}^{(h,1)}(-n, a+x/d) [d]_q^n}{d}. \hfill (3.17)$$

Thus, we see that the function \(L_{E,q,h}^{(h,1)}(s, x : \chi)\) interpolates generalized \((h,q)\)-Euler polynomials attached to \(\chi\) at negative integer values of \(s\) as followsings.

**Theorem 3.8.** For \(n \in \mathbb{N}\), let \(\chi\) be the Dirichlet’s character with odd conductor \(d\). Then one has

$$L_{E,q,h}^{(h,1)}(-n, x : \chi) = E_{n,q,h,\chi}^{(h,1)}(x). \hfill (3.18)$$

Note that if we take \(x = 1\), then Theorem 3.8 reduces to Theorem 3.3.
Let \(a\) and \(F\) be integers with \(F \equiv 1 \pmod{2}\) and \(0 < a < F\). For \(s \in \mathbb{C}\), we define partial \((h,q)\)-Hurwitz type zeta function \(H_{E,q}^{(h,1)}(s,a,x \mid F)\) as follows:

\[
H_{E,q}^{(h,1)}(s,a,x \mid F) = \sum_{m=a \pmod{F}}^{\infty} \frac{(-1)^{m} a^{h} m^{s} j}{[m + x]_{q}^{s}}.
\] (3.19)

By substituting \(m = a + jF\), we have

\[
H_{E,q}^{(h,1)}(s,a,x \mid F) = \sum_{j=0}^{\infty} \frac{(-1)^{a+jF} q^{h(a+jF)} x^{a+jF}}{[a+jF+x]_{q}^{s}}
\]  \[= (-1)^{a} q^{a} x^{a} [F]_{q}^{-s} \sum_{j=0}^{\infty} \frac{(-1)^{jF} (qF)^{j} (q^{F})^{j}}{[((a+x)/F) + j]_{q}^{s}}
\]  \[= [F]_{q}^{-s} \frac{(-1)^{a} q^{a} x^{a} s_{(h,1)}^{(h,1)}(s,a+x|F)}{1 + q^{F}}.
\] (3.20)

By substituting \((3.2)\), for \(s = -n\), we get

\[
H_{E,q}^{(h,1)}(s,a,x \mid F) = [F]_{q}^{-n} \frac{(-1)^{a} q^{a} x^{a}}{1 + q^{F}} E_{n,q}^{(h,1)}(a+x|F).
\] (3.21)

Equation (3.20) means that the function \(H_{E,q}^{(h,1)}(s,a,x \mid F)\) interpolates \(E_{n,q}^{(h,1)}(s,a,x \mid F)\) polynomials at negative integers.

From (3.16) and (3.20), we have the following theorem.

**Theorem 3.9.** For \(s \in \mathbb{C}\), \(\zeta \equiv 1 \pmod{2}\), let \(\chi \) be the Dirichlet’s character with conductor \(d \in \mathbb{N}\) with \(d \equiv 1 \pmod{2}\) and \(x \in \mathbb{R}\), \(0 < x \leq 1\), \(F\) is any multiple of \(d\). Then one has

\[
L_{E,q}^{(h,1)}(s,x : \chi) = (1 + q^{F}) \frac{1}{\chi(a)} \sum_{a=1}^{F} \chi(a)(-1)^{a} H_{E,q}^{(h,1)}(s,a,x \mid F).
\] (3.22)

**Remark 3.10.** If we take \(s = 0\) in (3.22), then we have

\[
L_{E,q}^{(h,1)}(s,x : \chi) = (1 + q^{F}) \frac{1}{\chi(a)} \sum_{a=1}^{F} \chi(a)(-1)^{a} H_{E,q}^{(h,1)}(s,a,x \mid F)
\]  \[= \frac{1}{1 + q^{F}} \sum_{a=1}^{F} \chi(a)(-1)^{a} q^{a} x^{a} \xi_{0,q}^{(h,1)}(a+x|F).
\] (3.23)

From (2.12), if we take \(s = 0\), then we have the following corollary.
Corollary 3.11. For \( s \in \mathbb{C}, \, q^r = 1 \) with \( \xi \neq 1 \), let \( \chi \) be the Dirichlet’s character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1 \) (mod 2) and \( x \in \mathbb{R}, \, 0 < x \leq 1 \), \( F \) is any multiple of \( d \). Then one has

\[
L_{E,q,d}^{(h,1)}(0, x : \chi) = \frac{(1 + q^2)^2}{(1 + q)(1 + \xi q^2)} \sum_{a=1}^{F} \chi(a)(-1)^a q^a \xi^a.
\] (3.24)

4. \( p \)-Adic Twisted Two-Variable Euler \((h, q)\)-L-Functions

In [62], Washington constructed one-variable \( p \)-adic-L-function which interpolates generalized classical Bernoulli numbers negative integers. Kim [22] investigated the \( p \)-adic analogues of two-variables Euler \( q \)-L-function. In this section, we will construct \( p \)-adic twisted two-variable Euler-(\( h, q) \)-L-functions, which interpolate generalized twisted \((h, q)\)-Euler polynomials at negative integers. Our notations and methods are essentially due to Kim and Washington (cf. [22, 62]). We assume that \( q \in \mathbb{C}_p \) with \( |1 - q|_p < p^{-(1/(p-1))} \), so that \( q^x = \exp(x \log q) \). Let \( p \) be an odd prime number. Let \( \omega \) denote the Teichmüller character having conductor \( p \). For an arbitrary character \( \chi \), we define \( \chi_n = \chi \omega^n \), where \( n \in \mathbb{Z} \), in the sense of the product of characters. Let \( \langle a \rangle = \langle a : q \rangle = \omega^{-1}(a)[a]_q = [a]_q/\omega(a) \). Then \( \langle a \rangle \equiv 1 \) (mod \( p^{1/(1/(p-1))} \)). Hence we see that

\[
\langle a + pt \rangle = \omega^{-1}(a + pt)[a + pt]_q
= \omega^{-1}(a)[a]_q + \omega^{-1}(a)q^a[p]_q
\equiv 1 \text{ (mod } p^{1/(1/(p-1))}\text{)},
\] (4.1)

where \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \), \( (a, p) = 1 \).

We denote the subset \( D \) of \( \mathbb{C}_p^* \) by (cf. [62])

\[
D = \{ s \in \mathbb{C}_p : |s|_p \leq p^{1-(1/(p-1))} \}. \tag{4.2}
\]

Let

\[
A_j(x) = \sum_{j=0}^{\infty} a_{n,j} x^n, \quad a_{n,j} \in \mathbb{C}_p, \, j = 0, 1, 2, \ldots, \tag{4.3}
\]

be a sequence of power series, each of which converges in a fixed subset \( D \) such that

(1) \( a_{n,j} \to a_{n,0} \) as \( j \to \infty \) for all \( n, j \) and

(2) for each \( s \in D \) and \( \epsilon > 0 \), there exists \( n_0 = n_0(s, \epsilon) \) such that

\[
\left| \sum_{p \geq n_0} a_{n,j} s^n \right|_p < \epsilon, \quad \text{for } \forall j. \tag{4.4}
\]

Then \( \lim_{j \to \infty} A_j(s) = A_0(s) \) for all \( s \in D \) (cf. [2, 22, 50, 51, 60, 62]).
Let \( \chi \) be the Dirichlet’s character with conductor \( d \) with \( d \equiv 1 \pmod{2} \) and let \( F \) be a positive multiple of \( p \) and \( d \).

Now we set

\[
L^{(h,1)}_{E,p,q,\xi}(s, x : \chi) = \frac{1 + q}{1 + q^d} \sum_{a=1, p \nmid a}^{F} \chi(a)(-1)^a \xi^a \langle a + pt \rangle^{-s}
\]

\[
\cdot \sum_{j=0}^{\infty} \left( -s \right)^j E^{(h,1)}_{j, q^f, \xi^j} q^{j(a+pt)} \left[ \frac{F}{a + pt} \right]_{q^f}^j.
\]

Then \( L^{(h,1)}_{E,p,q,\xi}(s, x : \chi) \) is analytic for \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \), when \( s \in D \). For \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \), we have

\[
\sum_{j=0}^{\infty} \left( -s \right)^j E^{(h,1)}_{j, q^f, \xi^j} q^{j(a+pt)} \left[ \frac{F}{a + pt} \right]_{q^f}^j
\]

is analytic for \( s \in D \). It readily follows that

\[
\langle a + pt \rangle^s = \omega^{-s}(a) [a + pt]^s_1 = \langle a \rangle^s \sum_{m=0}^{\infty} \left( \frac{s}{m} \right) \left( q^f [a]_q^{-1} [pt]_q \right)^m
\]

is analytic for \( s \in \mathbb{C}_p \) with \( |t|_p \leq 1 \) when \( s \in D \). Thus we see that

\[
L^{(h,1)}_{E,p,q,\xi}(0, x : \chi) = \frac{1 + q}{2} \sum_{a=1}^{F} (-1)^a \chi(n(a)) \xi^a.
\]

Let \( n \in \mathbb{Z}_+ \) and fixed \( t \in \mathbb{C}_p \) with \( |t|_p \leq 1 \). Then we have that

\[
E^{(h,1)}_{n \chi, \xi}(pt) = [F]^n_{q^f} \frac{1 + q^p}{1 + q^{E/p}} \sum_{a=0}^{F} \chi(n(a)) (-1)^a \xi^a E^{(h,1)}_{n \chi, \xi} \left( \frac{a + pt}{F} \right).
\]

If \( \chi(n(p)) \neq 0 \), then \((p, d_{\chi}) = 1\), so \( F/p \) is a multiple of \( d_{\chi} \). Therefore, we have

\[
\chi(n(p))[p]_{n \chi, \xi} E^{(h,1)}_{n \chi, \xi}(t)
\]

\[
= \chi(n(p))[p]_{n \chi, \xi} \left( [F]^n_{q^f} \frac{1 + q^p}{1 + q^{E/p}} \sum_{a=0}^{F/p-1} \chi(n(a)) (-1)^a \xi^a E^{(h,1)}_{n \chi, \xi} \left( \frac{a + pt}{F/p} \right) \right)
\]

\[
= [F]^n_{q^f} \frac{1 + q^p}{1 + q^{F/p}} \sum_{a=0}^{F} \chi(n(a)) (-1)^a \xi^a E^{(h,1)}_{n \chi, \xi} \left( \frac{a + pt}{F} \right).}
\]
Then we note that
\[
\frac{1 + q}{1 + q^p} x_n(p)[p]_q^n E^{(h,1)}_{n,q^a q^b, x_n} (t) = \frac{1 + q}{1 + q^p} \left[ F \right]_q^n \sum_{a=0 \atop p|a}^F x_n(a)(-1)^{a} q^{a} E^{(h,1)}_{n,q^a q^b} \left( \frac{a + pt}{F} \right).
\]
(4.11)

The difference of these equations yields
\[
E^{(h,1)}_{n,q^a q^b, x_n} (pt) - \frac{1 + q}{1 + q^p} x_n(p)[p]_q^n E^{(h,1)}_{n,q^a q^b, x_n} (t) = \frac{1 + q}{1 + q^p} \left[ F \right]_q^n \sum_{a=0 \atop p|a}^F x_n(a)(-1)^{a} q^{a} E^{(h,1)}_{n,q^a q^b} \left( \frac{a + pt}{F} \right).
\]
(4.12)

Using distribution for \((h,q)\)-Euler polynomials, we easily see that
\[
E^{(h,1)}_{n,q^a q^b} \left( \frac{a + pt}{F} \right) = \left[ F \right]_q^n \sum_{k=0}^n \binom{n}{k} q^{(a+pt)k} \sum_{a=0 \atop p|a}^F x_n(a)(-1)^{a} q^{a} E^{(h,1)}_{n,q^a q^b} \left( \frac{a + pt}{F} \right).
\]
(4.13)

Since \(x_n(a) = \chi(a) \omega^n(a)\), for \((a, p) = 1\), and \(t \in \mathbb{C}_p\), with \(|t|_p \leq 1\), we have
\[
E^{(h,1)}_{n,q^a q^b, x_n} (pt) - \frac{1 + q}{1 + q^p} x_n(p)[p]_q^n E^{(h,1)}_{n,q^a q^b, x_n} (t)
\]
\[
= \frac{1 + q}{1 + q^p} \sum_{a=0 \atop p|a}^F x_n(a)(-1)^{a} q^{a} E^{(h,1)}_{n,q^a q^b} \left( \frac{a + pt}{F} \right)
\]
\[
= \frac{1 + q}{1 + q^p} \sum_{a=0 \atop p|a}^F x_n(a)(-1)^{a} q^{a} (a + pt)^n \sum_{k=0}^n \binom{n}{k} q^{(a+pt)k} \left[ \frac{F}{a + pt} \right]_q^{k} E^{(h,1)}_{k,q^a q^b}.
\]
(4.14)

From (4.5)–(4.14), we can derive that
\[
E^{(h,1)}_{n,q^a q^b, x_n} (pt) - \frac{1 + q}{1 + q^p} x_n(p)[p]_q^n E^{(h,1)}_{n,q^a q^b, x_n} (t) = L^{(h,1)}_{E,p,q^b} (-n, t : \chi).
\]
(4.15)

Therefore we obtain the following theorem.

**Theorem 4.1.** Let \(F\) be a positive integral multiple of \(p\) and \(d (= d_{\chi})\) with \(F \equiv 1 \pmod{2}\), and let
\[
L^{(h,1)}_{E,p,q^b}\left(s, t : \chi\right) = \frac{1 + q}{1 + q^d} \sum_{a=1, \atop p|a}^F \chi(a)(-1)^{a} q^{a} (a + pt)^{-s} \sum_{m=0}^{\infty} \left( \frac{-s}{m} \right) q^{(a+pt)m} \left[ \frac{F}{a + pt} \right]_q^{m} E^{(h,1)}_{m,q^a q^b}.
\]
(4.16)
Furthermore, for each $n \in \mathbb{Z}_+$, we have
\begin{equation}
L_{E,p,q,h}^{(h_1)}(-n,t : \chi) = E_{n,q,h}^{(h_1)}(pt) - \frac{1 + q^n}{1 + q^n} \chi_n(p)[p]_q^n E_{n,q,h}^{(h_1)}(t) \tag{4.17}
\end{equation}

Thus we note that $L_{E,p,q,h}^{(h_1)}(s,0 : \chi) = L_{E,p,q,h}^{(h_1)}(s,\chi)$ for all $s \in D$, where $L_{E,p,q,h}^{(h_1)}(s,\chi)$ is twisted $p$-adic Euler $(h,q)$-L-function, (cf. [15, 22]).

We now generalized to two-variable $p$-adic Euler $(h,q)$-L-function, $L_{E,p,q,h}^{(h_1)}(s,t : \chi)$ which is first defined by the interpolation function
\begin{equation}
H_{E,p,q,h}^{(h_1)}(s,a,x | F) = \frac{(-1)^a}{1 + q^a} q^h \xi^a (a + pt)^{-s} \sum_{j=0}^{\infty} \left( \begin{array}{c} -s \\ j \end{array} \right) q^j (a + pt) \left( \frac{[F]_q}{[a + pt]_q} \right)^j E_{n,q,h}^{(h_1)}(t), \tag{4.18}
\end{equation}

for $s \in \mathbb{Z}_+$.

From (4.18), we have that
\begin{align}
H_{E,p,q,h}^{(h_1)}(-n,a,x | F) &= \frac{(-1)^a}{1 + q^a} q^h \xi^a (a + pt)^n \sum_{j=0}^{\infty} \left( \begin{array}{c} n \\ j \end{array} \right) q^j (a + pt) \left( \frac{[F]_q}{[a + pt]_q} \right)^j E_{n,q,h}^{(h_1)}(t) \\
&= \frac{(-1)^a}{1 + q^a} q^h \xi^a \omega^{-n}(a) [F]_q^n E_{n,q,h}^{(h_1)} \left( \frac{a}{F} \right) \\
&= \omega^{-n}(a) H_{E,q,h}^{(h_1)}(-n,a,x | F). \tag{4.19}
\end{align}

By using the definition of $H_{E,p,q,h}^{(h_1)}(s,a,x | F)$, we can express $L_{E,p,q,h}^{(h_1)}(s,t : \chi)$ for all $a \in \mathbb{Z}_+$, $(a,p) = 1$ and $t \in \mathbb{C}_p$ with $|t| \leq 1$ as follows:
\begin{equation}
L_{E,p,q,h}^{(h_1)}(s,t : \chi) = \sum_{a=1, p|a}^{F} \chi(a) H_{E,p,q,h}^{(h_1)}(s,a + pt | F). \tag{4.20}
\end{equation}

We know that $H_{E,p,q,h}^{(h_1)}(s,a + pt | F)$ is analytic for $t \in \mathbb{C}_p$, $|t| \leq 1$, when $s \in D$. The value of $(\partial/\partial s)L_{E,p,q,h}^{(h_1)}(s,t : \chi)$ is the coefficients of $s$ in the expansion of $L_{E,p,q,h}^{(h_1)}(s,t : \chi)$ at $s = 0$. Using the Taylor expansion at $s = 0$, we see that
\begin{equation}
(a + pt)^{-s} = 1 - s \log(a + pt) + \cdots, \quad \left( \begin{array}{c} -s \\ m \end{array} \right) = \frac{(-1)^m}{m} s + \cdots. \tag{4.21}
\end{equation}
The $p$-adic logarithmic function, $\log_p$, is the unique function $\mathbb{C}_p^* \rightarrow \mathbb{C}_p$ that satisfies

\[
\log_p(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n, \quad |x|_p < 1,
\]
\[
\log_p(xy) = \log_p(x) + \log_p(y), \quad \forall x, y \in \mathbb{C}_p^*,
\] (4.22)
\[
\log_p(p) = 0.
\]

By employing these expansion and some algebraic manipulations, we evaluate the derivative $(\partial / \partial s) L_{E,p,q,a}^{(h,1)} (0, t : \chi)$. It follows from the definition of $L_{E,p,q,a}^{(h,1)} (s, t : \chi)$ that

\[
L_{E,p,q,a}^{(h,1)} (s, t : \chi) = \frac{1 + q}{1 + q^F} \sum_{\substack{a=1 \atop p|a}}^{F} \chi(a) (1 - \frac{a}{p})^s (a + pt)^{-s}
\]
\[
\times \sum_{m=0}^{\infty} \left( -\frac{s}{m} \right) q^m \left( \frac{F}{a + pt} \right)^m E_{m,q,a}^{(h,1)}.
\] (4.23)

Thus, we have

\[
\left. \frac{\partial}{\partial s} L_{E,p,q,a}^{(h,1)} (s, t : \chi) \right|_{s=0} = \frac{1 + q}{1 + q^F} \sum_{\substack{a=1 \atop p|a}}^{F} \chi(a) (1 - \frac{a}{p})^s
\]
\[
\times \left( -\log(a + pt) \right) L_{E,0,q,a}^{(h,1)} + \sum_{m=1}^{\infty} \left( -\frac{s}{m} \right) q^m \left( \frac{F}{a + pt} \right)^m E_{m,q,a}^{(h,1)}.
\] (4.24)

Since $\omega(a)$ is a root of unity for $(a, p) = 1$, we have

\[
\log_p(a + pt) = \log_p(a + pt) + \log_p \omega^{-1}(a) = \log_p(a + pt).
\] (4.25)

Thus we have the following theorem.

**Theorem 4.2.** Let $\chi$ be a primitive Dirichlet’s character with odd conductor $d$, $d \in \mathbb{N}$ and let $F$ be a odd positive integral multiple of $p$ and $d$. Then for any $t \in \mathbb{C}_p$ with $|t| \leq 1$, one has

\[
\left. \frac{\partial}{\partial s} L_{E,p,q,a}^{(h,1)} (s, t : \chi) \right|_{s=0} = \frac{1 + q}{1 + q^F} \sum_{\substack{a=1 \atop p|a}}^{F} \chi(a) (1 - \frac{a}{p})^s \sum_{m=1}^{\infty} \left( -\frac{s}{m} \right) q^m \left( \frac{F}{a + pt} \right)^m E_{m,q,a}^{(h,1)}
\]
\[
- \frac{1 + q}{2} \sum_{\substack{a=1 \atop p|a}}^{F} \chi(a) (1 - \frac{a}{p})^s \log(a + pt).
\] (4.26)
References


