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Research Article

A General Iterative Process for Solving a System of Variational Inclusions in Banach Spaces

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The purpose of this paper is to introduce a general iterative method for finding solutions of a general system of variational inclusions with Lipschitzian relaxed cocoercive mappings. Strong convergence theorems are established in strictly convex and 2-uniformly smooth Banach spaces. Moreover, we apply our result to the problem of finding a common fixed point of a countable family of strict pseudo-contraction mappings.

1. Introduction

Let $U_E = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x + y}{2}\right\| \leq 1 - \delta. \tag{1.1}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U_E$. The norm of E is said to be Fréchet differentiable if, for any $x \in U_E$, the above limit is attained uniformly for all $y \in U_E$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\right\}, \tag{1.3}$$

where $\rho : [0,\infty) \to [0,\infty)$. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0}(\rho(\tau)/\tau) = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

From [1], we know the following property.

Let q be a real number with $1 < q \leq 2$ and let E be a Banach space. Then E is q-uniformly smooth if and only if there exists a constant $K \geq 1$ such that

$$\|x+y\|^q + \|x-y\|^q \leq 2\left(\|x\|^q + \|Ky\|^q\right), \quad \forall x, y \in E. \tag{1.4}$$

The best constant K in the above inequality is called the q-uniformly smoothness constant of E (see [1] for more details).

Let E be a real Banach space and E^* the dual space of E. Let $\langle \cdot,\cdot \rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{ f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\right\}, \quad \forall x \in E. \tag{1.5}$$

In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$. If E is a Hilbert space, then $J = I$ is the identity. Note the following.

(1) E is a uniformly smooth Banach space if and only if J is single-valued and uniformly continuous on any bounded subset of E.

(2) All Hilbert spaces, L^p (or l^p) spaces ($p \geq 2$), and the Sobolev spaces W_m^p ($p \geq 2$) are 2-uniformly smooth, while L^p (or l^p) and W_m^p spaces ($1 < p \leq 2$) are p-uniformly smooth.

(3) Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p,2\}$-uniformly smooth for any $p > 1$.

Further, we have the following properties of the generalized duality mapping J_q:

(i) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$,

(ii) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$,

(iii) $J_q(-x) = -J_q(x)$ for all $x \in E$.

It is known that, if X is smooth, then J is single valued. Recall that the duality mapping J is said to be weakly sequentially continuous if, for each sequence $\{x_n\} \subset E$ with $x_n \to x$ weakly, we have $J(x_n) \to J(x)$ weakly-$*$. We know that, if X admits a weakly sequentially continuous duality mapping, then X is smooth. For the details, see [2].

Let C be a nonempty closed convex subset of a smooth Banach space E. Recall the following definitions of a nonlinear mapping $\Psi : C \to E$, the following are mentioned.

Definition 1.1. Given a mapping $\Psi : C \to E$.

(i) Ψ is said to be *accretive* if

$$\langle \Psi x - \Psi y, J(x-y) \rangle \geq 0 \tag{1.6}$$

for all $x, y \in C$.

(ii) Ψ is said to be *α-strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle \Psi x - \Psi y, J(x-y) \rangle \geq \alpha \|x - y\|^2 \tag{1.7}$$

for all $x, y \in C$.

(iii) Ψ is said to be *α-inverse-strongly accretive* or *α-cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle \Psi x - \Psi y, J(x-y) \rangle \geq \alpha \|\Psi x - \Psi y\|^2 \tag{1.8}$$

for all $x, y \in C$.

(iv) Ψ is said to be *α-relaxed cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle \Psi x - \Psi y, J(x-y) \rangle \geq -\alpha \|\Psi x - \Psi y\|^2 \tag{1.9}$$

for all $x, y \in C$.

(v) Ψ is said to be *(α, β)-relaxed cocoercive* if there exist positive constants α and β such that

$$\langle \Psi x - \Psi y, J(x-y) \rangle \geq (-\alpha) \|\Psi x - \Psi y\|^2 + \beta \|x - y\|^2 \tag{1.10}$$

for all $x, y \in C$.

Remark 1.2. (1) Every α-strongly accretive mapping is an accretive mapping.

(2) Every α-strongly accretive mapping is a (β, α)-relaxed cocoercive mapping for any positive constant β but the converse is not true in general. Then the class of relaxed cocoercive operators is more general than the class of strongly accretive operators.

(3) Evidently, the definition of the inverse-strongly accretive operator is based on that of the inverse-strongly monotone operator in real Hilbert spaces (see, e.g., [3]).

(4) The notion of the cocoercivity is applied in several directions, especially for solving variational inequality problems using the auxiliary problem principle and projection methods [4]. Several classes of relaxed cocoercive variational inequalities have been studied in [5, 6].

Next, we consider a system of quasivariational inclusions as follows.

Find $(x^*, y^*) \in E \times E$ such that

$$\begin{aligned} &0 \in x^* - y^* + \rho_1 (\Psi_1 y^* + M_1 x^*), \\ &0 \in y^* - x^* + \rho_2 (\Psi_2 x^* + M_2 y^*), \end{aligned} \tag{1.11}$$

where $\Psi_i : E \to E$ and $M_i : E \to 2^E$ are nonlinear mappings for each $i = 1, 2$.

As special cases of problem (1.11), we have the following.

(1) If $\Psi_1 = \Psi_2 = \Psi$ and $M_1 = M_2 = M$, then problem (1.11) is reduced to the following.

Find $(x^*, y^*) \in E \times E$ such that

$$0 \in x^* - y^* + \rho_1(\Psi y^* + Mx^*),$$
$$0 \in y^* - x^* + \rho_2(\Psi x^* + My^*). \tag{1.12}$$

(2) Further, if $x^* = y^*$ in problem (1.12), then problem (1.12) is reduced to the following

Find $x^* \in E$ such that

$$0 \in \Psi x^* + Mx^*. \tag{1.13}$$

In 2006, Aoyama et al. [7] considered the following problem.

Find $u \in C$ such that

$$\langle \Psi u, J(v-u) \rangle \geq 0, \quad \forall v \in C. \tag{1.14}$$

They proved that the variational inequality (1.14) is equivalent to a fixed point problem. The element $u \in C$ is a solution of the variational inequality (1.14) if and only if $u \in C$ satisfies the following equation:

$$u = P_C(u - \lambda \Psi u), \tag{1.15}$$

where $\lambda > 0$ is a constant and P_C is a sunny nonexpansive retraction from E onto C, see the definition below.

Let D be a subset of C, and P be a mapping of C into D. Then P is said to be sunny if

$$P(Px + t(x - Px)) = Px, \tag{1.16}$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into itself is called a retraction if $P^2 = P$. If a mapping P of C into itself is a retraction, then $Pz = z$ for all $z \in R(P)$, where $R(P)$ is the range of P. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

The following results describe a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 1.3 (see [8]). *Let E be a smooth Banach space and C a nonempty subset of E. Let $P : E \to C$ be a retraction and J the normalized duality mapping on E. Then the following are equivalent:*

(1) *P is sunny and nonexpansive,*

(2) $\langle x - Px, J(y - Px) \rangle \leq 0$, *for all $x \in E$, $y \in C$.*

Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.17}$$

A mapping $T : C \to C$ is said to be *ε-strictly pseudocontractive* if there exists a constant $\varepsilon \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \varepsilon\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.18}$$

Note that the class of ε-strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \le \|x - y\|, \tag{1.19}$$

for all $x, y \in C$. That is, T is nonexpansive if and only if T is 0-strict pseudocontractive. We denote by $F(T) := \{x \in C : Tx = x\}$ the set of fixed points of T.

Proposition 1.4 (see [9]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and T a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

Definition 1.5. A countable family of mapping $\{T_n : C \to C\}_{i=1}^{\infty}$ is called a *family of uniformly ε-strict pseudocontractions* if there exists a constant $\varepsilon \in [0,1)$ such that

$$\|T_n x - T_n y\|^2 \le \|x - y\|^2 + \varepsilon\|(I - T_n)x - (I - T_n)y\|^2, \quad \forall x, y \in C,\ \forall n \ge 1. \tag{1.20}$$

For the class of nonexpansive mappings, one classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [10, 11]. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in C, \tag{1.21}$$

where $u \in C$ is a fixed point and $T : C \to C$ is a nonexpansive mapping. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C; that is,

$$x_t = tu + (1 - t)Tx_t. \tag{1.22}$$

It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Ceng et al. [12] proved that, if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [11] extended Browder's result to the setting of Banach spaces and proved that, if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T, and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Reich [11] showed that, if E is uniformly smooth and D is the fixed point set of a nonexpansive mapping from C into itself, then there is a unique sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Proposition 1.6 (see [11]). *Let E be a uniformly smooth Banach space and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. For each fixed $u \in C$ and every $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $P : C \to D$ by $Pu = s - \lim_{t\to 0} x_t$. Then P is the unique sunny nonexpansive retract from C onto D; that is, P satisfies the property.*

$$\langle u - Pu, J(y - Pu)\rangle \leq 0, \quad \forall u \in C,\ y \in D. \tag{1.23}$$

Notation. We use $Pu = s - \lim_{t\to 0} x_t$ to denote strong convergence to Pu of the net $\{x_t\}$ as $t \to 0$.

Definition 1.7 (see [13]). Let $M : E \to 2^E$ be a multivalued maximal accretive mapping. The single-valued mapping $J_{(M,\rho)} : E \to E$ defined by

$$J_{(M,\rho)}(u) = (I + \rho M)^{-1}(u), \quad \forall u \in E, \tag{1.24}$$

is called the resolvent operator associated with M, where ρ is any positive number and I is the identity mapping.

Recently, many authors have studied the problems of finding a common element of the set of fixed points of a nonexpansive mapping and one of the sets of solutions to the variational inequalities (1.11)–(1.14) by using different iterative methods (see, e.g., [7, 14–16]).

Very recently, Qin et al. [16] considered the problem of finding the solutions of a general system of variational inclusion (1.11) with α-inverse strongly accretive mappings. To be more precise, they obtained the following results.

Lemma 1.8 (see [16]). *For any $(x^*, y^*) \in E \times E$, where $y^* = J_{(M_2,\rho_2)}(x^* - \rho_2\Psi_2 x^*)$, (x^*, y^*) is a solution of the problem* (1.11) *if and only if x^* is a fixed point of the mapping Q defined by*

$$Q(x) = J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(x - \rho_2\Psi_2 x) - \rho_1\Psi_1 J_{(M_2,\rho_2)}(x - \rho_2\Psi_2 x)\right]. \tag{1.25}$$

Theorem QCCK (see [16, Theorem 2.1]). *Let E be a uniformly convex and 2-uniformly smooth Banach space with the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a γ_i-inverse-strongly accretive mapping, respectively, for each $i = 1,2$. Let $T : E \to E$ be a ε-strict pseudocontraction such that $F(T) \neq \emptyset$. Define a mapping S by $Sx = (1 - \varepsilon/K^2)x + (\varepsilon/K^2)Tx$, for all $x \in E$. Assume that $\Omega = F(T) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 1.8. Let $x_1 = u \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} z_n &= J_{(M_2,\rho_2)}(x_n - \rho_2\Psi_2 x_n), \\ y_n &= J_{(M_1,\rho_1)}(z_n - \rho_1\Psi_1 z_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n)\left[\mu S x_n + (1-\mu) y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{1.26}$$

where $\mu \in (0,1)$, $\rho_1 \in (0, \gamma_1/K^2]$, $\rho_2 \in (0, \gamma_2/K^2]$, *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$. *If the control consequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following restrictions:*

(C1) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

then $\{x_n\}$ *converges strongly to* $x^* = P_\Omega u$, *where* P_Ω *is the sunny nonexpansive retraction from* E *onto* Ω *and* (x^*, y^*), *where* $y^* = J_{(M_2,\rho_2)}(x^* - \rho_2\Psi_2 x^*)$, *is a solution to problem* (1.11).

On the other hand, we recall the following well-known definitions and results.

In a smooth Banach space, a mapping $A : C \to E$ is called *strongly positive* [17] if there exists a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, J(x)\rangle \geq \overline{\gamma}\|x\|^2, \|aI - bA\| = \sup_{\|x\|\leq 1}|\langle (aI - bA)x, J(x)\rangle|, \quad a \in [0,1],\ b \in [-1,1], \tag{1.27}$$

where I is the identity mapping and J is the normalized duality mapping.

In [18], Moudafi introduced the viscosity approximation method for nonexpansive mappings (see [19] for further developments in both Hilbert and Banach spaces). Let f be a contraction on C. Starting with an arbitrary initial point $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \quad n \geq 0 \tag{1.28}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved [18, 19] that, under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.28) strongly converges to the unique solution q in C of the variational inequality

$$\langle (I - f)q, p - q\rangle \geq 0, \quad \forall p \in C, \tag{1.29}$$

Recently, Marino and Xu [20] introduced the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n\gamma f(x_n), \quad n \geq 0. \tag{1.30}$$

where A is a strongly positive bounded linear operator on a Hilbert space H. They proved that, if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.30) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \tag{1.31}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle - h(x), \tag{1.32}$$

where h is a potential function for γf(i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Recently, Qin et al. [21] introduce the following iterative algorithm scheme:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= P_C\left[\beta_n + (1-\beta_n)Tx_n\right], \\ x_{n+1} &= \alpha_n\gamma f(x_n) + (1-\alpha_n A)y_n, \end{aligned} \tag{1.33}$$

where T is nonself-k-strict pseudo-contraction, f is a contraction, and A is a strongly positive bounded linear operator on a Hilbert space H. They proved, under certain appropriate conditions imposed on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ defined by (1.33) converges strongly to a fixed point of T, which solves some variational inequality.

In this paper, motivated by Qin et al. [16], Moudafi [18], Marino and Xu [20], and Qin et al. [21], we introduce a general iterative approximation method for finding common elements of the set of solutions to a general system of variational inclusions (1.11) with Lipschitzian and relaxed cocoercive mappings and the set common fixed points of a countable family of strict pseudocontractions. We prove the strong convergence theorems of such iterative scheme for finding a common element of such two sets which is a unique solution of some variational inequality and is also the optimality condition for some minimization problems in strictly convex and 2-uniformly smooth Banach spaces. The results presented in this paper improve and extend the corresponding results announced by Qin et al. [16], Moudafi [18], Marino and Xu [20], Qin et al. [21], and many others.

2. Preliminaries

Now we collect some useful lemmas for proving the convergence result of this paper.

Lemma 2.1 (see [22]). *The resolvent operator $J_{(M,\rho)}$ associated with M is single valued and nonexpansive for all $\rho > 0$.*

Lemma 2.2 (see [13]). *$u \in E$ is a solution of variational inclusion* (1.13) *if and only if $u = J_{(M,\rho)}(u-\rho\Psi u)$, for all $\rho > 0$; that is,*

$$VI(E,\Psi,M) = F\left(J_{(M,\rho)}(I-\rho\Psi)\right), \quad \forall\rho > 0, \tag{2.1}$$

where $VI(E,\Psi,M)$ denotes the set of solutions to problem (1.13).

Lemma 2.3 (see [23]). *Let E be a strictly convex Banach space. Let T_1 and T_2 be two nonexpansive mappings from E into itself with a common fixed point. Define a mapping S by*

$$Sx = \lambda T_1 x + (1-\lambda)T_2 x, \quad \forall x \in E, \tag{2.2}$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Lemma 2.4 (see [24]). *Let C be a nonempty closed convex subset of reflexive Banach space E which satisfies Opial's condition, and suppose that $T : C \to E$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ imply that $x = Tx$.*

Lemma 2.5 (see [25]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \tag{2.3}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.6 (see [26]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty} \left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \right) \leq 0. \tag{2.4}$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Definition 2.7 (see [27]). Let $\{S_n\}$ be a family of mappings from a subset C of a Banach space E into E with $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. We say that $\{S_n\}$ satisfies the AKTT-condition if, for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup_{z\in B} \|S_{n+1}z - S_n z\| < \infty. \tag{2.5}$$

Remark 2.8. The example of the sequence of mappings $\{S_n\}$ satisfying AKTT-condition is supported by Example 3.11.

Lemma 2.9 (see [27, Lemma 3.2]). *Suppose that $\{S_n\}$ satisfies AKTT-condition. Then, for each $y \in C$, $\{S_n y\}$ converses strongly to a point in C. Moreover, let the mapping S be defined by*

$$Sy = \lim_{n\to\infty} S_n y, \quad \forall y \in C. \tag{2.6}$$

Then for each bounded subset B of C, $\lim_{n\to\infty} \sup_{z\in B} \|Sz - S_n z\| = 0$.

Lemma 2.10 (see [28]). *Let E be a real 2-uniformly smooth Banach space and $T : E \to E$ a λ-strict pseudocontraction. Then $S := (1 - \lambda/K^2)I + \lambda/K^2 T$ is nonexpansive and $F(T) = F(S)$.*

Lemma 2.11 (see [29]). *Let E be a real 2-uniformly smooth Banach space with the best smoothness constant K. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E. \tag{2.7}$$

Lemma 2.12 (see [17, Lemma 1.8]). *Assume that A is a strongly positive linear bounded operator on a smooth Banach space E with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

3. Main Results

In this section, we prove that the strong convergence theorem for a countable family of uniformly ε-strict pseudocontractions in a strictly convex and 2-uniformly smooth Banach space admits a weakly sequentially continuous duality mapping. Before proving it, we need the following theorem.

Theorem 3.1 (see [17, Lemma 1.9]). *Let C be a nonempty closed convex subset of a reflexive, smooth Banach space E which admits a weakly sequentially continuous duality mapping J from E to E^*. Let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty, let $f : C \to C$ be a contraction with coefficient $\alpha \in (0, 1)$, and let A be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Then the net $\{x_t\}$ defined by*

$$x_t = t\gamma f(x_t) + (1 - tA)Tx_t \tag{3.1}$$

converges strongly as $t \to 0$ to a fixed point $\widetilde{x}$ of T which solves the variational inequality:

$$\langle (A - \gamma f)\widetilde{x}, J(\widetilde{x} - z)\rangle \leq 0, \quad \forall z \in F(T). \tag{3.2}$$

Lemma 3.2. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the smoothness constant K. Let $\Psi : C \to E$ be an L_Ψ-Lipschitzian and relaxed (c, d)-cocoercive mapping. Then*

$$\|(I - \lambda\Psi)x - (I - \lambda\Psi)y\|^2 \leq \left(1 + 2\lambda c L_\Psi^2 - 2\lambda d + 2\lambda^2 K^2 L_\Psi^2\right)\|x - y\|^2. \tag{3.3}$$

If $\lambda \leq (d - cL_\Psi^2)/K^2 L_\Psi^2$, then $I - \lambda\Psi$ is nonexpansive.

Proof. Using Lemma 2.11 and the cocoercivity of the mapping Ψ, we have, for all $x, y \in C$,

$$\begin{aligned}\|(I-\lambda\Psi)x-(I-\lambda\Psi)y\|^2 &= \|(x-y)-(\lambda\Psi x-\lambda\Psi y)\|^2\\&= \|x-y\|^2-2\lambda\langle\Psi x-\Psi y, J(x-y)\rangle+2\lambda^2K^2\|\Psi x-\Psi y\|^2\\&\le \|x-y\|^2-2\lambda\Big[-c\|\Psi x-\Psi y\|^2+d\|x-y\|^2\Big]+2\lambda^2K^2\|\Psi x-\Psi y\|^2\\&= \|x-y\|^2-2\lambda d\|x-y\|^2+2\lambda c\|\Psi x-\Psi y\|^2+2\lambda^2K^2\|\Psi x-\Psi y\|^2\\&\le \Big(1+2\lambda cL_\Psi^2-2\lambda d+2\lambda^2K^2L_\Psi^2\Big)\|x-y\|^2.\end{aligned} \tag{3.4}$$

Hence (3.3) is proved. Assume that $\lambda \le (d-cL_\Psi^2)/K^2L_\Psi^2$. Then, we have $(1+2\lambda cL_\Psi^2-2\lambda d+2\lambda^2K^2L_\Psi^2) \le 1$. This together with (3.3) implies that $I-\lambda\Psi$ is nonexpansive. □

Lemma 3.3. *Let E be a strictly convex and 2-uniformly smooth Banach space admiting a weakly sequentially continuous duality mapping with the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, (d_i - c_iL_i^2)/K^2L_i^2)$, respectively, for each $i = 1, 2$. Let $\{T_n : E \to E\}_{n=1}^{\infty}$ be a countable family of uniformly ε-strict pseudocontractions. Define a mapping $S_n : E \to E$ and $G_n : E \to E$ by*

$$\begin{gathered}S_nx=\left(1-\frac{\varepsilon}{K^2}\right)x+\frac{\varepsilon}{K^2}T_nx, \quad \forall x\in C,\ n\ge 1,\\G_n=\mu S_n+(1-\mu)Q,\end{gathered} \tag{3.5}$$

where Q is defined as in Lemma 1.8. Assume that $\Omega := \bigcap_{n=1}^{\infty}F(T_n)\cap F(Q)\neq\emptyset$. Let $f : E \to E$ be an α-contraction; let $A : E \to E$ be a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma}$ with $0<\gamma<\overline{\gamma}/\alpha$. Then the following hold.

(i) *For each $n \in \mathbb{N}$, G_n is nonexpansive such that*

$$F(G_n)=F(S_n)\cap F(Q)=F(T_n)\cap F(Q). \tag{3.6}$$

(ii) *Suppose that $\{G_n\}$ satisfies AKTT-condition. Let $G : E \to E$ be the mapping defined by $Gy = \lim_{n\to\infty}G_ny$ for all $y \in E$ and suppose that $F(G) = \bigcap_{n=1}^{\infty}F(G_n)$. The net $\{x_t\}$ defined by $x_t = t\gamma f(x_t)+(I-tA)Gx_t$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of G, which solves the variational inequality*

$$\langle (A-\gamma f)\tilde{x}, J(\tilde{x}-z)\rangle \le 0, \quad \forall z\in F(G) \tag{3.7}$$

and $(\tilde{x},\tilde{y})$ is a solution of general system of variational inequality problem (1.11) such that $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x}-\rho_2\Psi_2\tilde{x})$.

Proof . It follows from Lemma 2.10 that S_n is nonexpansive such that $F(S_n) = F(T_n)$ for each $n \in \mathbb{N}$. Next, we prove that Q is nonexpansive. Indeed, we observe that

$$\begin{aligned} Q(x) &= J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(x-\rho_2\Psi_2 x) - \rho_1\Psi_1 J_{(M_2,\rho_2)}(x-\rho_2\Psi_2 x)\right] \\ &= J_{(M_1,\rho_1)}(I-\rho_1\Psi_1)J_{(M_2,\rho_2)}(I-\rho_2\Psi_2)x. \end{aligned} \tag{3.8}$$

The nonexpansivity of $J_{(M_1,\rho_1)}$, $J_{(M_2,\rho_2)}$, $I-\rho_1\Psi_1$, and $I-\rho_2\Psi_2$ implies that Q is nonexpansive. By Lemma 2.3, we have that G_n is nonexpansive such that

$$F(G_n) = F(S_n) \cap F(Q) = F(T_n) \cap F(Q) \neq \emptyset, \quad \forall n \in \mathbb{N}. \tag{3.9}$$

Hence (i) is proved. It is well known that, if E is uniformly smooth, then E is reflexive. Hence Theorem 3.1 implies that $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of G, which solves the variational inequality

$$\left\langle (A-\gamma f)\tilde{x}, J(\tilde{x}-z)\right\rangle \leq 0, \quad \forall z \in F(G), \tag{3.10}$$

and $(\tilde{x},\tilde{y})$ is a solution of problem (1.11), where $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x}-\rho_2\Psi_2\tilde{x})$. This completes the proof of (ii). □

Theorem 3.4. *Let E be a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and has the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, (d_i - c_i L_i^2)/K^2 L_i^2)$, respectively, for each $i = 1, 2$. Let $\{T_n : E \to E\}_{n=1}^{\infty}$ be a countable family of uniformly ε-strict pseudocontractions. Define a mapping $S_n : E \to E$ by*

$$S_n x = \left(1-\frac{\varepsilon}{K^2}\right)x + \frac{\varepsilon}{K^2}T_n x, \quad \forall x \in C,\ n \geq 1. \tag{3.11}$$

Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 1.8. Let $f : E \to E$ be an α-contraction; let $A : E \to E$ be a strongly positive linear bounded self adjoint operator with coefficient $\overline{\gamma}$ with $0 < \gamma < \overline{\gamma}/\alpha$. Let $x_1 = u \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} z_n &= J_{(M_2,\rho_2)}(x_n - \rho_2\Psi_2 x_n), \\ y_n &= J_{(M_1,\rho_1)}(z_n - \rho_1\Psi_1 z_n), \\ x_{n+1} &= \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\left[\mu S_n x_n + (1-\mu)y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{3.12}$$

where $\mu \in (0,1)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that $\{S_n\}$ satisfies AKTT-condition. Let $S : E \to E$ be the mapping defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in E$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions

(C1) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$, *then* $\{x_n\}$ *converges strongly to* $\widetilde{x}$, *which solves the variational inequality*

$$\langle (A - \gamma f)\widetilde{x}, J(\widetilde{x} - z)\rangle \leq 0, \quad z \in \Omega, \tag{3.13}$$

and $(\widetilde{x}, \widetilde{y})$ *is a solution of general system of variational inequality problem* (1.11) *such that* $\widetilde{y} = J_{(M_2,\rho_2)}(\widetilde{x} - \rho_2\Psi_2\widetilde{x})$.

Proof. First, we show that sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded.

By the control condition (C2), we may assume, with no loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$.

Since A is a linear bounded operator on E, by (1.27), we have

$$\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in E, \|u\| = 1\}. \tag{3.14}$$

Observe that

$$\langle ((1-\beta_n)I - \alpha_n A)u, J(u)\rangle = 1 - \beta_n - \alpha_n\langle Au, J(u)\rangle \geq 1 - \beta_n - \alpha_n\|A\| \geq 0. \tag{3.15}$$

It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle ((1-\beta_n)I - \alpha_n A)u, J(u)\rangle : u \in E, \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Au, J(u)\rangle : u \in E, \|u\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n\overline{\gamma}.\end{aligned} \tag{3.16}$$

Therefore, taking $\overline{x} \in \Omega$, one has

$$\overline{x} = J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(\overline{x} - \rho_2\Psi_2\overline{x}) - \rho_1\Psi_1 J_{(M_2,\rho_2)}(\overline{x} - \rho_2\Psi_2\overline{x})\right]. \tag{3.17}$$

Putting $\overline{y} = J_{(M_2,\rho_2)}(\overline{x} - \rho_2\Psi_2\overline{x})$, one sees that

$$\overline{x} = J_{(M_1,\rho_1)}(\overline{y} - \rho_1\Psi_1\overline{y}). \tag{3.18}$$

It follows from Lemmas 2.1 and 3.2 that

$$\begin{aligned}\|z_n - \overline{y}\| &= \|J_{(M_2,\rho_2)}(x_n - \rho_2\Psi_2 x_n) - J_{(M_2,\rho_2)}(\overline{x} - \rho_2\Psi_2\overline{x})\| \\ &\leq \|(x_n - \rho_2\Psi_2 x_n) - (\overline{x} - \rho_2\Psi_2\overline{x})\| \\ &\leq \|x_n - \overline{x}\|.\end{aligned} \tag{3.19}$$

This implies that

$$\begin{aligned}\|y_n - \overline{x}\| &= \left\| J_{(M_1,\rho_1)}(z_n - \rho_1 \Psi_1 z_n) - J_{(M_1,\rho_1)}(\overline{y} - \rho_1 \Psi_1 \overline{y}) \right\| \\ &\le \left\| (z_n - \rho_1 \Psi_1 z_n) - (\overline{y} - \rho_1 \Psi_1 \overline{y}) \right\| \\ &\le \left\| z_n - \overline{y} \right\| \\ &\le \|x_n - \overline{x}\|.\end{aligned} \tag{3.20}$$

Setting $t_n = \mu S_n x_n + (1-\mu) y_n$ and applying Lemma 2.10, we have that S_n is a nonexpansive mapping such that $F(S_n) = F(T_n)$ for all $n \ge 1$ and hence $\bigcap_{n=1}^{\infty} F(S_n) = \bigcap_{n=1}^{\infty} F(T_n)$. Then

$$\begin{aligned}\|t_n - \overline{x}\| &= \left\| \mu S_n x_n + (1-\mu) y_n - \overline{x} \right\| \\ &\le \mu \|S_n x_n - \overline{x}\| + (1-\mu) \left\| y_n - \overline{x} \right\| \\ &\le \|x_n - \overline{x}\|.\end{aligned} \tag{3.21}$$

It follows from the last inequality that

$$\begin{aligned}\|x_{n+1} - \overline{x}\| &= \left\| \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) t_n - \overline{x} \right\| \\ &= \left\| \alpha_n (\gamma f(x_n) - A\overline{x}) + \beta_n (x_n - \overline{x}) + ((1-\beta_n) I - \alpha_n A)(t_n - \overline{x}) \right\| \\ &\le (1 - \beta_n - \alpha_n \overline{\gamma}) \|x_n - \overline{x}\| + \beta_n \|x_n - \overline{x}\| + \alpha_n \left\| \gamma f(x_n) - A\overline{x} \right\| \\ &\le (1 - \alpha_n \overline{\gamma}) \|x_n - \overline{x}\| + \alpha_n \gamma \alpha \|x_n - \overline{x}\| + \alpha_n \left\| \gamma f(\overline{x}) - A\overline{x} \right\| \\ &= (1 - \alpha_n (\overline{\gamma} - \gamma\alpha)) \|x_n - \overline{x}\| + \alpha_n \left\| \gamma f(\overline{x}) - A\overline{x} \right\|.\end{aligned} \tag{3.22}$$

By induction, we have

$$\|x_n - \overline{x}\| \le \max\left\{ \|x_1 - \overline{x}\|, \frac{\left\| \gamma f(\overline{x}) - A\overline{x} \right\|}{\overline{\gamma} - \gamma\alpha} \right\}, \quad n \ge 1. \tag{3.23}$$

This shows that the sequence $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{z_n\}$, and $\{t_n\}$.

On the other hand, from the nonexpansivity of the mappings $J_{(M_2,\rho_2)}$, one sees that

$$\begin{aligned}\left\| y_{n+1} - y_n \right\| &= \left\| J_{(M_1,\rho_1)}(z_{n+1} - \rho_1 \Psi_1 z_{n+1}) - J_{(M_1,\rho_1)}(z_n - \rho_1 \Psi_1 z_n) \right\| \\ &\le \left\| (z_{n+1} - \rho_1 \Psi_1 z_{n+1}) - (z_n - \rho_1 \Psi_1 z_n) \right\| \\ &\le \|z_{n+1} - z_n\|.\end{aligned} \tag{3.24}$$

In a similar way, one can obtain that

$$\|z_{n+1} - z_n\| \le \|x_{n+1} - x_n\|. \tag{3.25}$$

It follows that

$$\left\|y_{n+1}-y_n\right\| \leq \|x_{n+1}-x_n\|. \tag{3.26}$$

This implies that

$$\begin{aligned}
\|t_{n+1}-t_n\| &= \left\|\mu S_{n+1}x_{n+1}+(1-\mu)y_{n+1}-(\mu S_n x_n+(1-\mu)y_n)\right\| \\
&= \left\|\mu S_{n+1}x_{n+1}-\mu S_{n+1}x_n+(1-\mu)y_{n+1}+\mu S_{n+1}x_n-\mu S_n x_n-(1-\mu)y_n\right\| \\
&\leq \mu\|S_{n+1}x_{n+1}-S_{n+1}x_n\|+(1-\mu)\left\|y_{n+1}-y_n\right\|+\mu\|S_{n+1}x_n-S_n x_n\| \\
&\leq \mu\|x_{n+1}-x_n\|+(1-\mu)\|x_{n+1}-x_n\|+\mu\sup_{z\in\{x_n\}}\|S_{n+1}z-S_n z\| \\
&= \|x_{n+1}-x_n\|+\mu\sup_{z\in\{x_n\}}\|S_{n+1}z-S_n z\|.
\end{aligned} \tag{3.27}$$

Setting

$$x_{n+1}=(1-\beta_n)e_n+\beta_n x_n, \quad \forall n\geq 1, \tag{3.28}$$

one sees that

$$\begin{aligned}
e_{n+1}-e_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)t_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n A)t_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-At_{n+1})+t_{n+1}-\frac{\alpha_n}{1-\beta_n}(\gamma f(x_n)-At_n)-t_n,
\end{aligned} \tag{3.29}$$

and so it follows that

$$\|e_{n+1}-e_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|\gamma f(x_{n+1})-At_{n+1}\right\|+\frac{\alpha_n}{1-\beta_n}\left\|\gamma f(x_n)-At_n\right\|+\|t_{n+1}-t_n\|, \tag{3.30}$$

which, combined, with (3.27) yields that

$$\begin{aligned}
\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\| \leq{} & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|\gamma f(x_{n+1})-At_{n+1}\right\|+\frac{\alpha_n}{1-\beta_n}\left\|\gamma f(x_n)-At_n\right\| \\
&+\ \mu\sup_{z\in\{x_n\}}\|S_{n+1}z-S_n z\|.
\end{aligned} \tag{3.31}$$

Using the conditions (C1) and (C2) and AKTT-condition of $\{S_n\}$, we have

$$\limsup_{n\to\infty}(\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|) \leq 0. \tag{3.32}$$

Hence, from Lemma 2.6, it follows that

$$\lim_{n\to\infty} \|e_n - x_n\| = 0. \tag{3.33}$$

From (3.28), it follows that

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|e_n - x_n\|. \tag{3.34}$$

By (3.33), one sees that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.35}$$

On the other hand, one has

$$x_{n+1} - x_n = \alpha_n(\gamma f(x_n) - Ax_n) + ((1-\beta_n)I - \alpha_n A)(t_n - x_n). \tag{3.36}$$

It follows that

$$(1-\beta_n - \alpha_n\overline{\gamma})\|t_n - x_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - Ax_n\|. \tag{3.37}$$

From the conditions (C1), (C2) and from (3.35), one sees that

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{3.38}$$

Define the mapping G_n by

$$G_n = \mu S_n + (1-\mu)Q, \tag{3.39}$$

where Q is defined as in Lemma 1.8. From Lemma 3.3(i), we see that G_n is nonexpansive such that

$$F(G_n) = F(T_n) \cap F(Q) = F(S_n) \cap F(Q). \tag{3.40}$$

From (3.38), it follows that

$$\lim_{n\to\infty} \|G_n x_n - x_n\| = 0. \tag{3.41}$$

Since $\{S_n\}$ satisfies AKTT-condition and $S : E \to E$ is the mapping defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in E$, we have that $\{G_n\}$ satisfies AKTT-condition. Let the mapping $G : E \to E$ be the mapping defined by $Gy = \lim_{n\to\infty} G_n y$ for all $y \in E$. It follows from the nonexpansivity of S and

$$Gy = \mu Sy + (1-\mu)Q \tag{3.42}$$

that G is nonexpansive such that

$$F(G) = F(S) \cap F(Q) = \bigcap_{n=1}^{\infty} F(S_n) \cap F(Q) = \bigcap_{n=1}^{\infty} F(T_n) \cap F(Q) = \bigcap_{n=1}^{\infty} F(G_n). \tag{3.43}$$

Next, we prove that

$$\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_n - \tilde{x}) \rangle \le 0, \tag{3.44}$$

where $\tilde{x} = \lim_{t\to 0} x_t$ with x_t be the fixed point of the contraction

$$x \longmapsto t\gamma f(x) + (I - tA)Gx. \tag{3.45}$$

Then x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)Gx_t$. It follows from Lemma 3.3(ii) that $\tilde{x} \in F(G) = \Omega$, which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, J(\tilde{x} - z) \rangle \le 0, \quad \forall z \in F(G), \tag{3.46}$$

and $(\tilde{x}, \tilde{y})$ is a solution of general system of variational inequality problem (1.11) such that $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2 \Psi_2 \tilde{x})$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_{n_k} - \tilde{x}) \rangle = \limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_n - \tilde{x}) \rangle. \tag{3.47}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w \in C$ as $i \to \infty$. It follows from (3.41) and the nonexpansivity of G, we have $w \in F(G)$ by Lemma 2.4. Since the duality map J is single valued and weakly sequentially continuous from E to E^*, we get that

$$\begin{aligned}
\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_n - \tilde{x}) \rangle &= \lim_{k\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_{n_k} - \tilde{x}) \rangle \\
&= \lim_{i\to\infty} \left\langle \gamma f(\tilde{x}) - A\tilde{x}, J\left(x_{n_{k_i}} - \tilde{x}\right) \right\rangle \\
&= \langle (A - \gamma f)\tilde{x}, J(\tilde{x} - w) \rangle \le 0
\end{aligned} \tag{3.48}$$

as required. Now from Lemma 2.11, we have

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \left\|\alpha_n\gamma f(x_n)+\beta_n x_n+\left[(1-\beta_n)I-\alpha_n A\right]t_n-\tilde{x}\right\|^2\\
&= \left\|\left[(1-\beta_n)I-\alpha_n A\right](t_n-\tilde{x})+\alpha_n(\gamma f(x_n)-A\tilde{x})+\beta_n(x_n-\tilde{x})\right\|^2\\
&\le (1-\beta_n-\alpha_n\overline{\gamma})^2\|t_n-\tilde{x}\|^2+2\left\langle \alpha_n(\gamma f(x_n)-A\tilde{x})+\beta_n(x_n-\tilde{x}),J(x_{n+1}-\tilde{x})\right\rangle\\
&= (1-\beta_n-\alpha_n\overline{\gamma})^2\|t_n-\tilde{x}\|^2+2\beta_n\langle x_n-\tilde{x},J(x_{n+1}-\tilde{x})\rangle\\
&\quad+\ 2\alpha_n\left\langle\gamma f(x_n)-A\tilde{x},J(x_{n+1}-\tilde{x})\right\rangle\\
&= (1-\beta_n-\alpha_n\overline{\gamma})^2\|t_n-\tilde{x}\|^2+2\beta_n\langle x_n-\tilde{x},J(x_{n+1}-\tilde{x})\rangle\\
&\quad+\ 2\alpha_n\langle\gamma f(x_n)-\gamma f(\tilde{x}),J(x_{n+1}-\tilde{x})\rangle+2\alpha_n\left\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\right\rangle\\
&\le (1-\beta_n-\alpha_n\overline{\gamma})^2\|t_n-\tilde{x}\|^2+2\beta_n\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\|\\
&\quad+\ 2\alpha_n\left\|\gamma f(x_n)-\gamma f(\tilde{x})\right\|\|x_{n+1}-\tilde{x}\|+2\alpha_n\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\rangle\\
&\le (1-\beta_n-\alpha_n\overline{\gamma})^2\|x_n-\tilde{x}\|^2+\beta_n\left(\|x_{n+1}-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2\right)\\
&\quad+\ \alpha_n\gamma\alpha\left(\|x_{n+1}-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2\right)+2\alpha_n\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\rangle\\
&= \left[(1-\beta_n-\alpha_n\overline{\gamma})^2+\beta_n+\alpha_n\gamma\alpha\right]\|x_n-\tilde{x}\|^2+(\beta_n+\alpha_n\gamma\alpha)\|x_{n+1}-\tilde{x}\|^2\\
&\quad+\ 2\alpha_n\left\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\right\rangle,
\end{aligned}\tag{3.49}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\le \frac{(1-\beta_n-\alpha_n\overline{\gamma})^2+\beta_n+\alpha_n\gamma\alpha}{1-\beta_n-\alpha_n\gamma\alpha}\|x_n-\tilde{x}\|^2\\
&\quad+\frac{2\alpha_n}{1-\beta_n-\alpha_n\gamma\alpha}\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\rangle\\
&= \left[1-\frac{2\alpha_n(\overline{\gamma}-\gamma\alpha)}{1-\beta_n-\alpha_n\gamma\alpha}\right]\|x_n-\tilde{x}\|^2+\frac{\beta_n^2+2\beta_n\alpha_n\overline{\gamma}+\alpha_n^2\overline{\gamma}^2}{1-\beta_n-\alpha_n\gamma\alpha}\|x_n-\tilde{x}\|^2\\
&\quad+\ \frac{2\alpha_n}{1-\beta_n-\alpha_n\gamma\alpha}\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\rangle\\
&= \left[1-\frac{2\alpha_n(\overline{\gamma}-\gamma\alpha)}{1-\beta_n-\alpha_n\gamma\alpha}\right]\|x_n-\tilde{x}\|^2\\
&\quad+\frac{2\alpha_n(\overline{\gamma}-\gamma\alpha)}{1-\beta_n-\alpha_n\gamma\alpha}\left[\frac{\beta_n^2+2\beta_n\alpha_n\overline{\gamma}+\alpha_n^2\overline{\gamma}^2}{2\alpha_n(\overline{\gamma}-\gamma\alpha)}M_3+\ \frac{1}{\overline{\gamma}-\gamma\alpha}\left\langle\gamma f(\tilde{x})-A\tilde{x},J(x_{n+1}-\tilde{x})\right\rangle\right],
\end{aligned}\tag{3.50}$$

where M_3 is an appropriate constant such that $M_3 \geq \sup_{n\geq 0}\|x_n - \tilde{x}\|^2$. Put

$$j_n = \frac{2\alpha_n(\overline{\gamma} - \gamma\alpha)}{1 - \beta_n - \alpha_n\gamma\alpha}, \qquad k_n = \frac{\beta_n^2 + 2\beta_n\alpha_n\overline{\gamma} + \alpha_n^2\overline{\gamma}^2}{2\alpha_n(\overline{\gamma} - \gamma\alpha)}M_3 + \frac{1}{\overline{\gamma} - \gamma\alpha}\langle \gamma f(\tilde{x}) - A\tilde{x}, J(x_{n+1} - \tilde{x})\rangle, \tag{3.51}$$

that is,

$$\|x_{n+1} - \tilde{x}\|^2 \leq (1 - j_n)\|x_n - \tilde{x}\|^2 + j_n k_n. \tag{3.52}$$

It follows from conditions (C1), (C2) and from (3.44) that

$$\lim_{n\to\infty} j_n = 0, \qquad \sum_{n=1}^{\infty} j_n = \infty, \qquad \limsup_{n\to\infty} k_n \leq 0. \tag{3.53}$$

Apply Lemma 2.5 to (3.52) to conclude that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Setting $A \equiv I$, $\gamma = 1$, and $f := u$, we have the following result.

Theorem 3.5. *Let E be a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and has the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, (d_i - c_iL_i^2)/K^2L_i^2)$, respectively, for each $i = 1, 2$. Let $\{T_n : E \to E\}_{n=1}^{\infty}$ be a countable family of uniformly ε-strict pseudocontractions. Define a mapping $S_n : E \to E$ by*

$$S_n x = \left(1 - \frac{\varepsilon}{K^2}\right)x + \frac{\varepsilon}{K^2}T_n x, \quad \forall x \in C,\ n \geq 1. \tag{3.54}$$

Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 1.8. Let $x_1 = u \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} z_n &= J_{(M_2,\rho_2)}(x_n - \rho_2\Psi_2 x_n), \\ y_n &= J_{(M_1,\rho_1)}(z_n - \rho_1\Psi_1 z_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n)[\mu S_n x_n + (1 - \mu)y_n], \quad \forall n \geq 1, \end{aligned} \tag{3.55}$$

where $\mu \in (0,1)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that $\{S_n\}$ satisfies AKTT-condition. Let $S : E \to E$ be the mapping defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in E$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions

(C1) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

then $\{x_n\}$ converges strongly to $\tilde{x}$, which solves the variational inequality

$$\langle (I - f)\tilde{x}, J(\tilde{x} - z)\rangle \leq 0, \quad z \in \Omega, \tag{3.56}$$

and $(\tilde{x}, \tilde{y})$ *is a solution of general system of variational inequality problem* (1.11) *such that* $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2\Psi_2\tilde{x})$.

Remark 3.6. Theorem 3.4 mainly improves Theorem 2.1 of Qin et al. [16], in the following respects:

(a) from the class of inverse-strongly accretive mappings to the class of Lipchitzian and relaxed cocoercive mappings,

(b) from a ε-strict pseudocontraction to the countable family of uniformly ε-strict pseudocontractions,

(c) from a uniformly convex and 2-uniformly smooth Banach space to a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping.

Further, if $\{T_n : E \to E\}$ is a countable family of nonexpansive mappings, then Theorem 3.4 is reduced to the following result.

Theorem 3.7. *Let E be a strictly convex and* 2*-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and has the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, d_i - c_iL_i^2/K^2L_i^2)$, respectively, for each $i = 1, 2$. Let $\{T_n : E \to E\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings. Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap F(Q) \neq \emptyset$, where Q is defined as in* Lemma 1.8. *Let $f : E \to E$ be an α-contraction; let $A : E \to E$ be a strongly positive linear bounded self adjoint operator with coefficient $\overline{\gamma}$ with $0 < \gamma < \overline{\gamma}/\alpha$. Let $x_1 = u \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} z_n &= J_{(M_2,\rho_2)}(x_n - \rho_2\Psi_2 x_n), \\ y_n &= J_{(M_1,\rho_1)}(z_n - \rho_1\Psi_1 z_n), \\ x_{n+1} &= \alpha_n\gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)\left[\mu T_n x_n + (1-\mu)y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{3.57}$$

where $\mu \in (0,1)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that $\{T_n\}$ satisfies AKTT-condition. Let $T : E \to E$ be the mapping defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in E$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions

(C1) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

then $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, J(\tilde{x} - z)\rangle \leq 0, \quad z \in \Omega, \tag{3.58}$$

and $(\tilde{x}, \tilde{y})$ *is a solution of general system of variational inequality problem* (1.11) *such that* $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x} - \rho_2\Psi_2\tilde{x})$.

Remark 3.8. As in [27, Theorem 4.1], we can generate a sequence $\{T_n\}$ of nonexpansive mappings satisfying AKTT-condition; that is, $\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of E by using convex combination of a general sequence $\{S_k\}$ of nonexpansive mappings with a common fixed point. To be more precise, they obtained the following lemma.

Lemma 3.9 (see [27]). *Let C be a closed convex subset of a smooth Banach space E. Suppose that $\{S_k\}$ is a sequence of nonexpansive mappings of E into inself with a common fixed point. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by*

$$T_n x = \sum_{k=1}^{n} \beta_n^k S_k x, \quad \forall x \in E, \tag{3.59}$$

where $\{\beta_n^k\}$ is a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \le n$ such that

(i) $\sum_{k=1}^{n} \beta_n^k = 1$ *for all* $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \beta_n^k > 0$ *for every* $k \in \mathbb{N}$,

(iii) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$.

Then the following are given.

(1) *Each T_n is a nonexpansive mapping.*

(2) *$\{T_n\}$ satisfies AKTT-condition.*

(3) *If $T : C \to C$ is defined by*

$$Tx = \sum_{k=1}^{\infty} \beta_n^k S_k x, \quad \forall x \in C, \tag{3.60}$$

then $Tx = \lim_{n\to\infty} T_n x$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k)$.

Theorem 3.10. *Let E be a strictly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and has the smoothness constant K. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $\Psi_i : E \to E$ a L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, (d_i - c_i L_i^2)/K^2 L_i^2)$, respectively, for each $i = 1, 2$. Let $\{S_k : E \to E\}_{k=1}^{\infty}$ be a countable family of nonexpansive mappings. Assume that $\Omega := \bigcap_{k=1}^{\infty} F(S_k) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 1.8. Let $f : E \to E$ be an α-contraction; let $A : E \to E$ be a strongly positive linear bounded self adjoint operator with coefficient $\overline{\gamma}$ with $0 < \gamma < \overline{\gamma}/\alpha$. Let $x_1 = u \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} z_n &= J_{(M_2,\rho_2)}(x_n - \rho_2 \Psi_2 x_n), \\ y_n &= J_{(M_1,\rho_1)}(z_n - \rho_1 \Psi_1 z_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\left[\mu \sum_{k=1}^{n} \beta_n^k S_k x_n + (1-\mu) y_n\right], \quad \forall n \ge 1, \end{aligned} \tag{3.61}$$

where $\{\beta_n^k\}$ satisfies conditions (i)–(iii) of Lemma 3.9, $\mu \in (0,1)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that $\{T_n\}$ satisfies AKTT-condition. Let $T : E \to E$ be the mapping defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in E$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If the control consequences

$\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions:

(C1) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

then $\{x_n\}$ converges strongly to $\tilde{x}$, which solves the variational inequality

$$\langle (A-\gamma f)\tilde{x}, J(\tilde{x}-z)\rangle \le 0, \quad z \in \Omega, \tag{3.62}$$

and $(\tilde{x},\tilde{y})$ is a solution of general system of variational inequality problem (1.11) *such that $\tilde{y} = J_{(M_2,\rho_2)}(\tilde{x}-\rho_2\Psi_2\tilde{x})$.*

Proof. We write the iteration (3.61) as

$$\begin{aligned} z_n &= J_{(M_2,\rho_2)}(x_n - \rho_2\Psi_2 x_n), \\ y_n &= J_{(M_1,\rho_1)}(z_n - \rho_1\Psi_1 z_n), \\ x_{n+1} &= \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)[\mu T_n x_n + (1-\mu)y_n], \quad \forall n \ge 1, \end{aligned} \tag{3.63}$$

where T_n is defined by (3.59). It is clear that each mapping T_n is nonexpansive. By Theorem 3.7 and Lemma 3.9, the conclusion follows. □

The following example appears in [27] shows that there exists $\{\beta_n^k\}$ satisfying the conditions of Lemma 3.9.

Example 3.11. Let $\{\beta_n^k\}$ be defined by

$$\beta_n^k = \begin{cases} 2^{-k} & (k < n), \\ 2^{1-k} & (k = n), \end{cases} \tag{3.64}$$

for all $n, k \in \mathbb{N}$ with $k \le n$. In this case, the sequence $\{T_n\}$ of mappings generated by $\{S_k\}$ is defined as follows: For $x \in C$.

$$\begin{aligned} T_1 x &= S_1 x, \\ T_2 x &= \frac{1}{2}S_1 x + \frac{1}{2}S_2 x, \\ T_3 x &= \frac{1}{2}S_1 x + \frac{1}{4}S_2 x + \frac{1}{4}S_3 x, \\ T_4 x &= \frac{1}{2}S_1 x + \frac{1}{4}S_2 x + \frac{1}{8}S_3 x + \frac{1}{8}S_4 x, \\ &\vdots \\ T_n x &= \frac{1}{2}S_1 x + \frac{1}{4}S_2 x + \frac{1}{8}S_3 x + \frac{1}{16}S_4 x + \cdots + \frac{1}{2^{n-1}}S_{n-1} x + \frac{1}{2^{n-1}}S_n x. \end{aligned} \tag{3.65}$$

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References

[1] K. Ball, E. A. Carlen, and E. H. Lieb, "Sharp uniform convexity and smoothness inequalities for trace norms," *Inventiones Mathematicae*, vol. 115, no. 3, pp. 463–482, 1994.

[2] J.-P. Gossez and E. Lami Dozo, "Some geometric properties related to the fixed point theory for nonexpansive mappings," *Pacific Journal of Mathematics*, vol. 40, pp. 565–573, 1972.

[3] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[4] R. U. Verma, "Approximation-solvability of a class of A-monotone variational inclusion problems," *Journal of the Korean Society for Industrial and Applied Mathematics*, vol. 8, no. 1, pp. 55–66, 2004.

[5] R. U. Verma, "General over-relaxed proximal point algorithm involving A-maximal relaxed monotone mappings with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e1461–e1472, 2009.

[6] R. U. Verma, "A-monotonicity and applications to nonlinear variational inclusion problems," *Journal of Applied Mathematics and Stochastic Analysis*, no. 2, pp. 193–195, 2004.

[7] K. Aoyama, H. Iiduka, and W. Takahashi, "Weak convergence of an iterative sequence for accretive operators in Banach spaces," *Fixed Point Theory and Applications*, vol. 2006, Article ID 35390, 13 pages, 2006.

[8] S. Reich, "Asymptotic behavior of contractions in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 44, pp. 57–70, 1973.

[9] S. Kitahara and W. Takahashi, "Image recovery by convex combinations of sunny nonexpansive retractions," *Topological Methods in Nonlinear Analysis*, vol. 2, no. 2, pp. 333–342, 1993.

[10] F. E. Browder, "Fixed-point theorems for noncompact mappings in Hilbert space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 53, pp. 1272–1276, 1965.

[11] S. Reich, "Strong convergence theorems for resolvents of accretive operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.

[12] L.-C. Ceng, C.-Y. Wang, and J.-C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[13] S.-S. Zhang, J. H. W. Lee, and C. K. Chan, "Algorithms of common solutions to quasi variational inclusion and fixed point problems," *Applied Mathematics and Mechanics*, vol. 29, no. 5, pp. 571–581, 2008.

[14] Y. J. Cho, Y. Yao, and H. Zhou, "Strong convergence of an iterative algorithm for accretive operators in Banach spaces," *Journal of Computational Analysis and Applications*, vol. 10, no. 1, pp. 113–125, 2008.

[15] Y. Hao, "Strong convergence of an iterative method for inverse strongly accretive operators," *Journal of Inequalities and Applications*, vol. 2008, Article ID 420989, 9 pages, 2008.

[16] X. Qin, S. S. Chang, Y. J. Cho, and S. M. Kang, "Approximation of solutions to a system of variational inclusions in Banach spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 916806, 16 pages, 2010.

[17] G. Cai and C. S. Hu, "Strong convergence theorems of a general iterative process for a finite family of λ_i-strict pseudo-contractions in q-uniformly smooth Banach spaces," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 149–160, 2010.

[18] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[19] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[20] G. Marino and H.-K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.

[21] X. Qin, M. Shang, and S. M. Kang, "Strong convergence theorems of modified Mann iterative process for strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 3, pp. 1257–1264, 2009.

[22] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "On a strongly nonexpansive sequence in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 8, no. 3, pp. 471–489, 2007.

[23] R. E. Bruck Jr., "Properties of fixed-point sets of nonexpansive mappings in Banach spaces," *Transactions of the American Mathematical Society*, vol. 179, pp. 251–262, 1973.

[24] J. S. Jung, "Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 509–520, 2005.

[25] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

[26] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[27] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2350–2360, 2007.

[28] H. Zhou, "Convergence theorems for Λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 3160–3173, 2008.

[29] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.