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Research Article

A New Iteration Method for Nonexpansive Mappings and Monotone Mappings in Hilbert Spaces

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We introduce a new composite iterative scheme by the viscosity approximation method for nonexpansive mappings and monotone mappings in a Hilbert space. It is proved that the sequence generated by the iterative scheme converges strongly to a common point of set of fixed points of nonexpansive mapping and the set of solutions of variational inequality for an inverse-strongly monotone mappings, which is a solution of a certain variational inequality. Our results substantially develop and improve the corresponding results of [Chen et al. 2007 and Iiduka and Takahashi 2005]. Essentially a new approach for finding the fixed points of nonexpansive mappings and solutions of variational inequalities for monotone mappings is provided.

1. Introduction

Let H be a real Hilbert space and C a nonempty closed convex subset of H. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0,1)$ such that $\|f(x)-f(y)\| \le k\|x-y\|$, $x, y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. That is $\Sigma_C = \{f : C \to C \mid f$ is a contraction with constant $k\}$. A mapping $S : C \to C$ is called *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|$, $x, y \in C$; see [1, 2] for the results of nonexpansive mappings. We denote by $F(S)$ the set of fixed points of S; that is, $F(S) = \{x \in C : x = Sx\}$.

Let P_C be the metric projection of H onto C. A mapping A of C into H is called *monotone* if for $x,\ y \in C$, $\langle x - y, Ax - Ay\rangle \ge 0$. The *variational inequality problem* is to find a $u \in C$ such that

$$\langle v - u, Au\rangle \ge 0 \tag{1.1}$$

for all $v \in C$; see [3–6]. The set of solutions of the variational inequality is denoted by $\mathrm{VI}(C, A)$. A mapping A of C into H is called *inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \tag{1.2}$$

for all $x, y \in C$; see [7–9]. For such a case, A is called α-inverse-strongly monotone.

In 2005, Iiduka and Takahashi [10] introduced an iterative scheme for finding a common point of the set of fixed points of a nonexapnsive mapping and the set of solutions of the variational inequality for an inverse-strong monotone mapping as follows. For an α-inverse-strongly monotone mapping A of C to H and a nonexpansive mapping S of C into itself such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$, $x_1 = x \in C$, $\{\alpha_n\} \subset [0, 1)$, and $\{\lambda_n\} \subset [0, 2\alpha]$,

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n) \tag{1.3}$$

for every $n \geq 1$. They proved that the sequence generated by (1.3) converges strongly to $P_{F(S)\cap \mathrm{VI}(C,A)} x$ under the conditions on $\{\alpha_n\}$ and $\{\lambda_n\} : \lambda_n \in [c, d]$ for some c, d with $0 < c < d < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n < \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty. \tag{1.4}$$

On the other hand, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [11]. In 2004, in order to extend Theorem 2.2 of Moudafi [11] to a Banach space setting, Xu [12] considered the the following explicit iterative process. For $S : C \to C$ nonexpansive mappings, $f \in \sum_C$ and $\alpha_n \in (0, 1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S x_n, \quad n \geq 1. \tag{1.5}$$

Moreover, in [12], he also studied the strong convergence of $\{x_n\}$ generated by (1.5) as $n \to \infty$ in either a Hilbert space or a uniformly smooth Banach space and showed that the strong $\lim_{n\to\infty} x_n$ is a solution of a certain variational inequality.

In 2007, Chen et al. [13] considered the following iterative scheme as the viscosity approximation method of (1.3). For an α-inverse-strongly-monotone mapping A of C to H and a nonexpansive mapping S of C into itself such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$, $f \in \Sigma_C$, $x_0 \in C$, $\{\alpha_n\} \subset [0, 1)$, and $\{\lambda_n\} \subset [0, 2\alpha]$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \quad n \geq 0, \tag{1.6}$$

and showed that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a point in $F(S) \cap \mathrm{VI}(C, A)$ under condition (1.4) on $\{\alpha_n\}$ and $\{\lambda_n\}$, which is a solution of a certain variational inequality.

In this paper, motivated by above-mentioned results, we introduce a new composite iterative scheme by the viscosity approximation method. For an α-inverse-strongly monotone mapping A of C to H and a nonexpansive mapping S of C into itself such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$, $f \in \Sigma_C$, $x_0 \in C$, $\{\alpha_n\} \subset [0,1)$, and $\{\lambda_n\} \subset [0, 2\alpha]$,

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= (1-\beta_n) y_n + \beta_n S P_C(y_n - \lambda_n A y_n), \quad n \geq 0. \end{aligned} \tag{1.7}$$

If $\beta_n = 0$, then the iterative scheme (1.7) reduces to the iterative scheme (1.6). Under condition (1.4) on the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ and appropriate condition on sequence $\{\beta_n\}$, we show that the sequence $\{x_n\}$ generated by (1.7) converges strongly to a point in $F(S) \cap \mathrm{VI}(C, A)$, which is a solution of a certain variational inequality. Using this result, we also obtain a strong convergence result for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping. Moreover, we investigate the problem of finding a common point of the set of fixed points of a nonexpansive mapping and the set of zeros of an inverse-strongly monotone mapping. The main results develop and improve the corresponding results of Chen et al. [13] and Iiduka and Takahashi [10]. We point out that the iterative scheme (1.7) is a new approach for finding the fixed points of nonexpansive mappings and solutions of variational inequalities for monotone mappings.

2. Preliminaries and Lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and C a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \tag{2.1}$$

for all $y \in C$. P_C is called the *metric projection* of H to C. It is well known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{2.2}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the properties

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \tag{2.3}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.4}$$

for all $x \in H$, $y \in C$. In the context of the variational inequality problem, this implies that

$$u \in \mathrm{VI}(C, A) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \text{for any } \lambda > 0. \tag{2.5}$$

We state some examples for inverse-strongly monotone mappings. If $A = I - T$, where T is a nonexpansive mapping of C into itself and I is the identity mapping of H, then A is $1/2$-inverse-strongly monotone and $\mathrm{VI}(C, A) = F(T)$. A mapping A of C into H is called *strongly monotone* if there exists a positive real number η such that

$$\langle x - y, Ax - Ay \rangle \geq \eta \|x - y\|^2 \tag{2.6}$$

for all $x, y \in C$. In such a case, we say that A is η-strongly monotone. If A is η-strongly monotone and κ-*Lipschitz continuous*, that is, $\|Ax - Ay\| \leq \kappa \|x - y\|$ for all $x, y \in C$, then A is η/κ^2-inverse-strongly monotone.

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $1/\alpha$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay \rangle + \lambda^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \end{aligned} \tag{2.7}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H. The following result for the existence of solutions of the variational inequality problem for inverse-strongly monotone mappings was given in Takahashi and Toyoda [14].

Proposition 2.1. *Let C be a bounded closed convex subset of a real Hilbert space and A an α-inverse-strongly monotone mapping of C into H. Then, $VI(C, A)$ is nonempty.*

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the *normal cone* to C at v, that is, $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{2.8}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [15, 16].

We need the following lemmas for the proof of our main results.

Lemma 2.2 (see [17]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n, \quad n \geq 0, \tag{2.9}$$

where $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions:*

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$;

(ii) $\limsup_{n\to\infty} \beta_n/\lambda_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (see [1], demiclosedness principle). *Let* H *be a real Hilbert space,* C *a nonempty closed convex subset of* H*, and* $T : C \to E$ *a nonexpansive mapping. Then the mapping* $I - T$ *is demiclosed on* C*, where* I *is the identity mapping; that is,* $x_n \rightharpoonup x$ *in* E *and* $(I-T)x_n \to y$ *imply that* $x \in C$ *and* $(I-T)x = y$.

Lemma 2.4. *In a real Hilbert space* H*, there holds the following inequality:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.10}$$

for all $x, y \in H$.

3. Main Results

In this section, we introduce a new composite iterative scheme for nonexpansive mappings and inverse-strongly monotone mappings and prove a strong convergence of this scheme.

Theorem 3.1. *Let* C *be a closed convex subset of a real Hilbert space* H*. Let* A *be an* α*-inverse-strongly monotone mapping of* C *to* H *and* S *a nonexpansive mapping of* C *into itself such that* $F(S) \cap VI(C,A) \neq \emptyset$*, and* $f \in \Sigma_C$*. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} &x_0 \in C, \\ &y_n = \alpha_n f(x_n) + (1-\alpha_n) SP_C(x_n - \lambda_n A x_n), \\ &x_{n+1} = (1-\beta_n) y_n + \beta_n SP_C(y_n - \lambda_n A y_n), \quad n \geq 0, \end{aligned} \tag{3.1}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0,1)$*, and* $\{\beta_n\} \subset [0,1]$*. If* $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0,1)$;

(iii) $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 2\alpha$;

(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

then $\{x_n\}$ *converges strongly to* $q \in F(S) \cap VI(C,A)$*, which is a solution of the following variational inequality:*

$$\langle (I-f)(q), q-p \rangle \leq 0, \quad p \in F(S) \cap VI(C,A). \tag{3.2}$$

Proof. Let $z_n = P_C(x_n - \lambda_n A x_n)$ and $w_n = P_C(y_n - \lambda_n A y_n)$ for every $n \geq 0$. Let $u \in F(S) \cap \mathrm{VI}(C, A)$. Since $I - \lambda_n A$ is nonexpansive and $u = P_C(u - \lambda_n A u)$ from (2.5), we have

$$\begin{aligned} \|z_n - u\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(u - \lambda_n A u)\| \\ &\leq \|(x_n - \lambda_n A x_n) - (u - \lambda_n A u)\| \\ &\leq \|x_n - u\|. \end{aligned} \tag{3.3}$$

Similarly we have $\|w_n - u\| \leq \|y_n - u\|$.

We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, since

$$\begin{aligned} \|y_n - u\| &= \|\alpha_n(f(x_n) - u) + (1-\alpha_n)(Sz_n - u)\| \\ &\leq \alpha_n\|f(x_n) - u\| + (1-\alpha_n)\|z_n - u\| \\ &\leq \alpha_n\|f(x_n) - f(u)\| + \alpha_n\|f(u) - u\| + (1-\alpha_n)\|x_n - u\| \\ &\leq \alpha_n k\|x_n - u\| + (1-\alpha_n)\|x_n - u\| + \alpha_n\|f(u) - u\| \\ &= (1-(1-k)\alpha_n)\|x_n - u\| + \alpha_n\|f(u) - u\| \\ &\leq \max\left\{\|x_n - u\|, \frac{1}{1-k}\|f(u) - u\|\right\}, \end{aligned} \tag{3.4}$$

we have

$$\begin{aligned} \|x_{n+1} - u\| &= \|(1-\beta_n)\ (y_n - u) + \beta_n(Sw_n - u)\| \\ &\leq (1-\beta_n)\|y_n - u\| + \beta_n\|w_n - u\| \\ &\leq (1-\beta_n)\|y_n - u\| + \beta_n\|y_n - u\| \\ &\leq \max\left\{\|x_n - u\|, \frac{1}{1-k}\|f(u) - u\|\right\}. \end{aligned} \tag{3.5}$$

By induction, we get

$$\|x_n - u\| \leq \max\left\{\|x_0 - u\|, \frac{1}{1-k}\|f(u) - u\|\right\}, \quad n \geq 0. \tag{3.6}$$

This implies that $\{x_n\}$ is bounded and so $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{Ax_n\}$, and $\{Ay_n\}$ are bounded. Moreover, since $\|Sz_n - u\| \leq \|x_n - u\|$ and $\|Sw_n - u\| \leq \|y_n - u\|$, $\{Sz_n\}$ and $\{Sw_n\}$ are also bounded. By condition (i), we also obtain

$$\|y_n - Sz_n\| = \alpha_n\|f(x_n) - Sz_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.7}$$

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From (3.1), we have

$$y_n = \alpha_n f(x_n) + (1-\alpha_n)Sz_n,$$
$$y_{n-1} = \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})Sz_{n-1}, \quad n \ge 1. \tag{3.8}$$

Simple calculations show that

$$y_n - y_{n-1} = (1-\alpha_n)(Sz_n - Sz_{n-1}) + (\alpha_n - \alpha_{n-1})\big(f(x_{n-1}) - Sz_{n-1}\big) + \alpha_n\big(f(x_n) - f(x_{n-1})\big). \tag{3.9}$$

Since

$$\begin{aligned}\|z_n - z_{n-1}\| &\le \|(x_n - \lambda_n Ax_n) - (x_{n-1} - \lambda_{n-1}Ax_{n-1})\| \\ &\le \|(x_n - \lambda_n Ax_n) - (x_{n-1} - \lambda_n Ax_{n-1})\| + |\lambda_{n-1} - \lambda_n|\|Ax_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\lambda_{n-1} - \lambda_n|\|Ax_{n-1}\|\end{aligned} \tag{3.10}$$

for every $n \ge 1$, we have

$$\begin{aligned}\big\|y_n - y_{n-1}\big\| &\le (1-\alpha_n)\|z_n - z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - Sz_{n-1}\big\| + \alpha_n k\|x_n - x_{n-1}\| \\ &\le (1-\alpha_n)(\|x_n - x_{n-1}\| + |\lambda_{n-1} - \lambda_n|\|Ax_{n-1}\|) \\ &\quad + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - Sz_{n-1}\big\| + \alpha_n k\|x_n - x_{n-1}\| \\ &\le (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + L_1|\lambda_{n-1} - \lambda_n| + M_1|\alpha_n - \alpha_{n-1}|\end{aligned} \tag{3.11}$$

for every $n \ge 1$, where $M_1 = \sup\{\|f(x_n) - Sz_{n-1}\| : n \ge 1\}$ and $L_1 = \sup\{\|Ax_n\| : n \ge 0\}$.

On the other hand, from (3.1) we have

$$x_{n+1} = (1-\beta_n)y_n + \beta_n Sw_n,$$
$$x_n = (1-\beta_{n-1})y_{n-1} + \beta_{n-1}Sw_{n-1}. \tag{3.12}$$

Also, simple calculations show that

$$x_{n+1} - x_n = (1-\beta_n)(y_n - y_{n-1}) + \beta_n(Sw_n - Sw_{n-1}) + (\beta_n - \beta_{n-1})(Sw_{n-1} - y_{n-1}). \tag{3.13}$$

Since

$$\begin{aligned}\|w_n - w_{n-1}\| &\le \big\|(y_n - \lambda_n Ay_n) - (y_{n-1} - \lambda_{n-1}Ay_{n-1})\big\| \\ &\le \big\|(y_n - \lambda_n Ay_n) - (y_{n-1} - \lambda_n Ay_{n-1})\big\| + |\lambda_{n-1} - \lambda_n|\big\|Ay_{n-1}\big\| \\ &\le \big\|y_n - y_{n-1}\big\| + |\lambda_{n-1} - \lambda_n|\big\|Ay_{n-1}\big\|\end{aligned} \tag{3.14}$$

for every $n \geq 1$, it follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-\beta_n)\|y_n-y_{n-1}\|+\beta_n\|w_n-w_{n-1}\|+|\beta_n-\beta_{n-1}|\|Sw_{n-1}-y_{n-1}\| \\ &\leq (1-\beta_n)\|y_n-y_{n-1}\|+\beta_n(\|y_n-y_{n-1}\|+|\lambda_{n-1}-\lambda_n|\|Ay_{n-1}\|) \\ &\quad +|\beta_n-\beta_{n-1}|\|Sw_{n-1}-y_{n-1}\| \\ &\leq \|y_n-y_{n-1}\|+|\lambda_{n-1}-\lambda_n|\|Ay_{n-1}\|+|\beta_n-\beta_{n-1}|\|Sw_{n-1}-y_{n-1}\|.\end{aligned} \tag{3.15}$$

Substituting (3.11) into (3.15), we derive

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-(1-k)\alpha_n)\|x_n-x_{n-1}\|+L_1|\lambda_{n-1}-\lambda_n|+M_1|\alpha_n-\alpha_{n-1}| \\ &\quad +|\lambda_{n-1}-\lambda_n|\|Ay_{n-1}\|+|\beta_n-\beta_{n-1}|\|Sw_{n-1}-y_{n-1}\| \\ &\leq (1-(1-k)\alpha_n)\|x_n-x_{n-1}\|+L_2|\lambda_{n-1}-\lambda_n|+M_1|\alpha_n-\alpha_{n-1}|+M_2|\beta_n-\beta_{n-1}|,\end{aligned} \tag{3.16}$$

where $L_2 = \sup\{L_1+\|Ay_n\| : n \geq 1\}$ and $M_2 = \sup\{\|Sw_n-y_n\| : n \geq 0\}$. From conditions (i) and (iv), it is easy to see that

$$\begin{gathered}\lim_{n\to\infty}(1-k)\alpha_n = 0, \qquad \sum_{n=0}^{\infty}(1-k)\alpha_n = \infty, \\ \sum_{n=0}^{\infty}\left(M_1|\alpha_{n+1}-\alpha_n|+M_2|\beta_{n+1}-\beta_n|+L_2|\lambda_{n+1}-\lambda_n|\right) < \infty.\end{gathered} \tag{3.17}$$

Applying Lemma 2.2 to (3.16), we have

$$\|x_{n+1}-x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.18}$$

By (3.11), we also have that $\|y_{n+1}-y_n\| \to 0$ as $n \to \infty$.

Step 3. We show that $\lim_{n\to\infty}\|x_n-y_n\| = 0$ and $\lim_{n\to\infty}\|x_n-Sz_n\| = 0$. Indeed, it follows that

$$\begin{aligned}\|x_{n+1}-y_n\| &= \beta_n\|Sw_n-y_n\| \\ &\leq \beta_n(\|Sw_n-Sz_n\|+\|Sz_n-y_n\|) \\ &\leq a(\|w_n-z_n\|+\|Sz_n-y_n\|) \\ &\leq a(\|y_n-x_n\|+\|Sz_n-y_n\|) \\ &\leq a(\|y_n-x_{n+1}\|+\|x_{n+1}-x_n\|+\|Sz_n-y_n\|),\end{aligned} \tag{3.19}$$

which implies that

$$\|x_{n+1}-y_n\| \leq \frac{a}{1-a}(\|x_{n+1}-x_n\|+\|Sz_n-y_n\|). \tag{3.20}$$

Obviously, by (3.7) and Step 2, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. This implies that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.21}$$

By (3.7) and (3.21), we also have

$$\|x_n - Sz_n\| \le \|x_n - y_n\| + \|y_n - Sz_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.22}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - z_n\| = 0$. To this end, let $u \in F(S) \cap \mathrm{VI}(C, A)$. Then, by convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|y_n - u\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)Sz_n - u\|^2 \\
&\le \alpha_n\|f(x_n) - u\|^2 + (1-\alpha_n)\|Sz_n - u\|^2 \\
&\le \alpha_n\|f(x_n) - u\|^2 + (1-\alpha_n)\|z_n - u\|^2 \\
&\le \alpha_n\|f(x_n) - u\|^2 + (1-\alpha_n)\left[\|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Au\|^2\right] \\
&\le \alpha_n\|f(x_n) - u\|^2 + \|x_n - u\|^2 + (1-\alpha_n)c(d-2\alpha)\|Ax_n - Au\|^2.
\end{aligned} \tag{3.23}$$

So we obtain

$$\begin{aligned}
&-(1-\alpha_n)c(d-2\alpha)\|Ax_n - Au\|^2 \\
&\quad\le \alpha_n\|f(x_n) - u\|^2 + \left(\|x_n - u\| + \|y_n - u\|\right)\left(\|x_n - u\| - \|y_n - u\|\right) \\
&\quad\le \alpha_n\|f(x_n) - u\|^2 + \left(\|x_n - u\| + \|y_n - u\|\right)\|x_n - y_n\|.
\end{aligned} \tag{3.24}$$

Since $\alpha_n \to 0$ and $\|x_n - y_n\| \to 0$ by condition (i) and (3.21), we have $\|Ax_n - Au\| \to 0$ $(n \to \infty)$. Moreover, from (2.2) we obtain

$$\begin{aligned}
\|z_n - u\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\|^2 \\
&\le \langle x_n - \lambda_n Ax_n - (u - \lambda_n Au), z_n - u\rangle \\
&= \frac{1}{2}\Big\{\|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\|^2 + \|z_n - u\|^2 \\
&\qquad -\|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au) - (z_n - u)\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|x_n - u\|^2 + \|z_n - u\|^2 - \|x_n - z_n\|^2 \\
&\qquad +2\lambda_n\langle x_n - z_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2\Big\},
\end{aligned} \tag{3.25}$$

and so

$$\|z_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \langle x_n - z_n, Ax_n - Au\rangle - \lambda_n^2 \|Ax_n - Au\|^2. \tag{3.26}$$

And hence

$$\begin{aligned}\left\|y_n - u\right\|^2 &\le \alpha_n \left\|f(x_n) - u\right\|^2 + (1-\alpha_n)\|z_n - u\|^2 \\ &\le \alpha_n \left\|f(x_n) - u\right\|^2 + \|x_n - u\|^2 - (1-\alpha_n)\|x_n - z_n\|^2 \\ &\quad + 2(1-\alpha_n)\lambda_n \langle x_n - z_n, Ax_n - Au\rangle - (1-\alpha_n)\lambda_n^2 \|Ax_n - Au\|^2.\end{aligned} \tag{3.27}$$

Then we have

$$\begin{aligned}(1-\alpha_n)\|x_n - z_n\|^2 &\le \alpha_n \left\|f(x_n) - u\right\|^2 + \left(\|x_n - u\| + \left\|y_n - u\right\|\right)\left(\|x_n - u\| - \left\|y_n - u\right\|\right) \\ &\quad + 2(1-\alpha_n)\lambda_n \langle x_n - z_n, Ax_n - Au\rangle - (1-\alpha_n)\lambda_n^2 \|Ax_n - Au\|^2 \\ &\le \alpha_n \left\|f(x_n) - u\right\|^2 + \left(\|x_n - u\| + \left\|y_n - u\right\|\right)\left\|x_n - y_n\right\| \\ &\quad + 2(1-\alpha_n)\lambda_n \langle x_n - z_n, Ax_n - Au\rangle - (1-\alpha_n)\lambda_n^2 \|Ax_n - Au\|^2.\end{aligned} \tag{3.28}$$

Since $\alpha_n \to 0$, $\|x_n - y_n\| \to 0$ and $\|Ax_n - Au\| \to 0$, we get $\|x_n - z_n\| \to 0$. Also by (3.21), we have

$$\left\|y_n - z_n\right\| \le \left\|y_n - x_n\right\| + \|x_n - z_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.29}$$

Step 5. We show that $\limsup_{n\to\infty}\langle f(q) - q, y_n - q\rangle \le 0$ for $q \in F(S) \cap \mathrm{VI}(C, A)$, where q is a solution of the variational inequality

$$\left\langle (I - f)(q), q - p\right\rangle \le 0, \qquad p \in F(S) \cap \mathrm{VI}(C, A). \tag{3.30}$$

To this end, choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle f(q) - q, Sz_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, Sz_{n_i} - q\rangle. \tag{3.31}$$

Since $\{z_{n_i}\}$ is bounded, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to z. We may assume without loss of generality that $z_{n_i} \rightharpoonup z$. Since $\|Sz_{n_i} - z_{n_i}\| \le \|Sz_{n_i} - x_{n_i}\| + \|x_{n_i} - z_{n_i}\| \to 0$ by Steps 4 and 5, we have $Sz_{n_i} \rightharpoonup z$. Then we can obtain $z \in F(S) \cap \mathrm{VI}(C, A)$. Indeed, let us first show that $z \in \mathrm{VI}(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{3.32}$$

Then T is maximal monotone. Let $(v,w) \in G(T)$. Since $w - Av \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n, w - Av \rangle \geq 0. \tag{3.33}$$

On the other hand, from $z_n = P_C(x_n - \lambda_n A x_n)$, we have $\langle v - z_n, z_n - (x_n - \lambda_n A z_n) \rangle \geq 0$ and hence

$$\left\langle v - z_n, \frac{z_n - x_n}{\lambda_n} + A x_n \right\rangle \geq 0. \tag{3.34}$$

Therefore we have

$$\begin{aligned}
\langle v - z_{n_i}, w \rangle &\geq \langle v - z_{n_i}, Av \rangle \\
&\geq \langle v - z_{n_i}, Av \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} + A x_{n_i} \right\rangle \\
&= \left\langle v - z_{n_i}, Av - A x_{n_i} - \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - z_{n_i}, Av - A z_{n_i} \rangle + \langle v - z_{n_i}, A z_{n_i} - A x_{n_i} \rangle \\
&\quad - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - z_{n_i}, A z_{n_i} - A x_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle.
\end{aligned} \tag{3.35}$$

Hence we have $\langle v - z, w \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in \mathrm{VI}(C, A)$.

On the another hand, by Steps 3 and 4, $\|z_n - S z_n\| \leq \|z_n - x_n\| + \|x_n - S z_n\| \to 0$. So, by Lemma 2.3, we obtain $z \in F(S)$ and hence $z \in F(S) \cap \mathrm{VI}(C, A)$. Then by (3.30) we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(q) - q, S z_{n_i} - q \rangle = \lim_{i\to\infty} \langle f(q) - q, S z_{n_i} - q \rangle &= \langle f(q) - q, z - q \rangle \\
&= \langle (I - f)(q), q - z \rangle \leq 0.
\end{aligned} \tag{3.36}$$

Thus, from (3.7) we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(q) - q, y_n - q \rangle &\leq \limsup_{n\to\infty} \langle f(q) - q, y_n - S z_n \rangle + \limsup_{n\to\infty} \langle f(q) - q, S z_n - q \rangle \\
&\leq \limsup_{n\to\infty} \|f(q) - q\| \|y_n - S z_n\| + \limsup_{n\to\infty} \langle f(q) - q, S z_n - q \rangle \\
&\leq 0.
\end{aligned} \tag{3.37}$$

Step 6. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$ for $q \in F(S) \cap \mathrm{VI}(C, A)$, where q is a solution of the variational inequality

$$\langle (I - f)(q), q - p\rangle \leq 0, \quad p \in F(S) \cap \mathrm{VI}(C, A). \tag{3.38}$$

Indeed, from Lemma 2.4, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 \leq \|y_n - q\|^2 &= \|\alpha_n(f(x_n) - q) + (1-\alpha_n)(Sz_n - q)\|^2 \\
&\leq (1-\alpha_n)\|Sz_n - q\|^2 + 2\alpha_n\langle f(x_n) - q, y_n - q\rangle \\
&\leq (1-\alpha_n)^2\|z_n - q\|^2 + 2\alpha_n\langle f(x_n) - f(q), y_n - q\rangle + 2\alpha_n\langle f(q) - q, y_n - q\rangle \\
&\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n k\|x_n - q\|\|y_n - q\| + 2\alpha_n\langle f(q) - q, y_n - q\rangle \\
&\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n k\|x_n - q\|(\|y_n - x_n\| + \|x_n - q\|) \\
&\quad + 2\alpha_n\langle f(q) - q, y_n - q\rangle \\
&= (1 - 2(1-k)\alpha_n)\|x_n - q\|^2 + 2\alpha_n k\|y_n - x_n\|\|x_n - q\| + 2\alpha_n\langle f(q) - q, y_n - q\rangle \\
&\leq (1 - \overline{\alpha_n})\|x_n - q\|^2 + \overline{\alpha_n}\overline{\beta_n},
\end{aligned} \tag{3.39}$$

where

$$\overline{\alpha}_n = 2(1-k)\alpha_n, \qquad \overline{\beta}_n = \frac{kB}{1-k}\|y_n - x_n\| + \frac{1}{1-k}\langle f(q) - q, y_n - q\rangle, \tag{3.40}$$

and $B = \sup\{\|x_n - q\| : n \geq 0\}$. It is easily seen that $\overline{\alpha}_n \to 0$, $\sum_{n=1}^{\infty}\overline{\alpha}_n = \infty$, and $\limsup_{n\to\infty}\overline{\beta}_n \leq 0$. Thus by Lemma 2.2, we obtain $x_n \to q$. This completes the proof. □

Remark 3.2. (1) Theorem 3.1 improves the corresponding results in Chen et al. [13] and Iiduka and Takahashi [10]. In particular, if $\beta_n = 0$ and $f(x_n) = x$ is constant in (3.1), then Theorem 3.1 reduces to Theorem 3.1 of Iiduka and Takahashi [10].

(2) We obtain a new composite iterative scheme for a nonexpansive mapping if $A = 0$ in Theorem 3.1 as follows (see also Jung [18]):

$$\begin{aligned}
&x_0 \in C, \\
&y_n = \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \\
&x_{n+1} = (1-\beta_n)y_n + \beta_n Sy_n, \quad n \geq 0.
\end{aligned} \tag{3.41}$$

As a direct consequence of Theorem 3.1, we have the following result.

Corollary 3.3. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C to H such that $VI(C,A) \neq \emptyset$, and $f \in \Sigma_C$. Let $\{x_n\}$ be a sequence generated by*

$$
\begin{gathered}
x_0 \in C, \\
y_n = \alpha_n f(x_n) + (1-\alpha_n) P_C(x_n - \lambda_n A x_n), \\
x_{n+1} = (1-\beta_n) y_n + \beta_n P_C(y_n - \lambda_n A y_n), \quad n \geq 0,
\end{gathered}
\tag{3.42}
$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0,1)$, and $\{\beta_n\} \subset [0,1]$. If $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0,1)$,

(iii) $\lambda_n \in [c,d]$ *for some* c, d *with* $0 < c < d < 2\alpha$,

(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

then $\{x_n\}$ converges strongly to $q \in VI(C,A)$, which is a solution of the following variational inequality:

$$
\langle (I-f)(q), q-p \rangle \leq 0, \quad p \in VI(C,A). \tag{3.43}
$$

4. Applications

In this section, as in [10, 13], we obtain two theorems in a Hilbert space by using Theorem 3.1.

A mapping $T : C \to C$ is called *strictly pseudocontractive* if there exists α with $0 \leq \alpha < 1$ such that

$$
\|Tx - Ty\|^2 \leq \|x-y\|^2 + \alpha \|(I-T)x - (I-T)y\|^2 \tag{4.1}
$$

for every $x, y \in C$. If $\alpha = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with α. Then A is $(1-\alpha)/2$-inverse-strongly monotone; see [7]. Actually, we have, for all $x, y \in C$,

$$
\|(I-A)x - (I-A)y\|^2 \leq \|x-y\|^2 + \alpha \|Ax - Ay\|^2. \tag{4.2}
$$

On the other hand, since H is a real Hilbert space, we have

$$
\|(I-A)x - (I-A)y\|^2 = \|x-y\|^2 + A\|Ax - Ay\|^2 - 2\langle x-y, Ax - Ay \rangle. \tag{4.3}
$$

Hence we have

$$
\langle x-y, Ax-Ay \rangle \geq \frac{1-\alpha}{2} \|Ax - Ay\|^2. \tag{4.4}
$$

Using Theorem 3.1, we first get a strong convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H. Let T be an α-strictly pseudocontractive mapping of C into itself and S a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$, and $f \in \Sigma_C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} &x_0 \in C, \\ &y_n = \alpha_n f(x_n) + (1-\alpha_n) S((1-\lambda_n)x_n + \lambda_n T x_n), \\ &x_{n+1} = (1-\beta_n) y_n + \beta_n S((1-\lambda_n) y_n + \lambda_n T y_n), \quad n \geq 0, \end{aligned} \tag{4.5}$$

where $\{\lambda_n\} \subset [0, 1-\alpha)$, $\{\alpha_n\} \subset [0,1)$, and $\{\beta_n\} \subset [0,1]$. If $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ satisfy the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0,1)$,

(iii) $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 1-\alpha$,

(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

then $\{x_n\}$ converges strongly to $q \in F(S) \cap F(T)$, which is a solution of the following variational inequality:

$$\langle (I-f)(q), q-p \rangle \leq 0, \quad p \in F(S) \cap F(T). \tag{4.6}$$

Proof. Put $A = I - T$. Then A is $(1-\alpha)/2$-inverse-strongly monotone. We have $F(T) = \mathrm{VI}(C, A)$ and $P_C(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n$. Thus, the desired result follows from Theorem 3.1. □

Using Theorem 3.1, we also have the following result.

Theorem 4.2. *Let H be a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of H into itself and S a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$, and $f \in \Sigma_C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} &x_0 \in H, \\ &y_n = \alpha_n f(x_n) + (1-\alpha_n) S(x_n - \lambda_n A x_n), \\ &x_{n+1} = (1-\beta_n) y_n + \beta_n S(y_n - \lambda_n A y_n), \quad n \geq 0, \end{aligned} \tag{4.7}$$

where $\{\lambda_n\} \subset [0, 2\alpha)$, $\{\alpha_n\} \subset [0,1)$, and $\{\beta_n\} \subset [0,1]$. If $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ satisfy the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0,1)$,

(iii) $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 2\alpha$,

(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, *then* $\{x_n\}$ *converges strongly to* $q \in F(S) \cap A^{-1}0$*, which is a solution of the following variational inequality:*

$$\langle (I - f)(q), q - p \rangle \leq 0, \quad p \in F(S) \cap A^{-1}0. \tag{4.8}$$

Proof. We have $A^{-1}0 = VI(H, A)$. So, putting $P_H = I$, by Theorem 3.1, we obtain the desired result. □

Remark 4.3. If $\beta_n = 0$ in Theorems 4.1 and 4.2, then Theorems 4.1 and 4.2 reduce to Chen et al. [13, Theorems 4.1 and 4.2]. Theorems 4.1 and 4.2 also extend in Iiduka and Takahashi [10, Theorems 4.1 and 4.2] to the viscosity methods in composite iterative schemes.

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