Research Article

Further Results on the Reverse Order Law for \( \{1,3\}\)-Inverse and \( \{1,4\}\)-Inverse of a Matrix Product

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Both Djordjević (2007) and Takane et al. (2007) have studied the equivalent conditions for \( B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\} \) and \( B\{1,4\}A\{1,4\} \subseteq (AB)\{1,4\} \). In this note, we derive the necessary and sufficient conditions for \( B\{1,3\}A\{1,3\} \geq (AB)\{1,3\} \), \( B\{1,4\}A\{1,4\} \geq (AB)\{1,4\} \), \( B\{1,3\}A\{1,3\} = (AB)\{1,3\} \) and \( B\{1,4\}A\{1,4\} = (AB)\{1,4\} \).

1. Introduction

Let \( \mathbb{C}^{m \times n} \) denote the set of all \( m \times n \) matrices over the complex field \( \mathbb{C} \). For \( A \in \mathbb{C}^{m \times n} \), its range space, null space, rank, and conjugate transpose will be denoted by \( \mathcal{R}(A) \), \( \mathcal{N}(A) \), \( r(A) \), and \( A^* \), respectively. The symbol \( \dim \mathcal{R}(A) \) denotes the dimension of \( \mathcal{R}(A) \). The \( n \times n \) identity matrix is denoted by \( I_n \), and if the size is obvious from the context, then the subscript on \( I_n \) can be neglected.

For a matrix \( A \in \mathbb{C}^{m \times n} \), a generalized inverse \( X \) of \( A \) is a matrix which satisfies some of the following four Penrose equations:

\[
(1) \ AXA = A, \quad (2) \ XAX = X, \quad (3) \ (AX)^* = AX, \quad (4) \ (XA)^* = XA. \tag{1.1}
\]

Let \( \emptyset \neq \eta \subseteq \{1,2,3,4\} \). Then \( A\eta \) denotes the set of all matrices \( X \) which satisfy (i) for all \( i \in \eta \). Any matrix \( X \in A\eta \) is called an \( \eta \)-inverse of \( A \). One usually denotes any \( \{1\}\)-inverse of \( A \) by \( A^{(1)} \) or \( A^\dagger \), and any \( \{1,3\}\)-inverse of \( A \) by \( A^{(1,3)} \) which is also called a least squares g-inverses of \( A \). Any \( \{1,4\}\)-inverse of \( A \) is denoted by \( A^{(1,4)} \) which is also called a minimum norm g-inverses of \( A \). The unique \( \{1,2,3,4\}\)-inverse of \( A \) is denoted by \( A^* \), which is called the Moore-Penrose generalized inverse of \( A \). General properties of the above generalized inverses can be found in [1–3]. The research in this area is active, especially about the \( \{2\}\)-inverse and the reverse order law for generalized inverse; see [4–7].
There are very good results for the reverse order law for \( \{1\} \)-inverse and \( \{1,2\} \)-inverse of two-matrix or multi-matrix products, and Liu and Yang \([8]\) studied equivalent conditions for \( B[1,3,4]A[1,3,4] \subseteq (AB)[1,3,4] \), \( B[1,3,4]A[1,3,4] \supseteq (AB)[1,3,4] \), and \( B[1,3,4]A[1,3,4] = (AB)[1,3,4] \). Moreover, Wei and Guo \([9]\) derived the reverse order law for \( \{1,3\} \)-inverse and \( \{1,4\} \)-inverse of two-matrix products by using the product singular value decomposition (P-SVD). However, there is a fly in the ointment in Wei and Guo’s results. That is, those results contain the information of subblock produced by P-SVD. In other words, they are related to P-SVD. In order to overcome this shortcoming, two methods are employed. One is operator theory; the other is maximal and minimal rank of matrix expressions. Using these two different methods, both \([6,10]\) obtain

\[
B[1,3]A[1,3] \subseteq (AB)[1,3] \iff \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B),
\]

\[
B[1,4]A[1,4] \subseteq (AB)[1,4] \iff \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*).
\]

These results are our hope because there is no information of the P-SVD in them. Note that \( \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \) and \( \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \) are equivalent to \( r(B,A^*AB) = r(B) \) and \( r(A^*,BB^*A^*) = r(A) \), respectively. Therefore, these results are only related to the range space (or the rank) of \( A, A^*, B, B^* \) or their expressions. However, there are no analogs for \( B[1,3]A[1,3] \supseteq (AB)[1,3] \) and \( B[1,4]A[1,4] \supseteq (AB)[1,4] \). In this note, we derive the necessary and sufficient conditions for them. And after this we present a new equivalent conditions for \( B[1,3]A[1,3] = (AB)[1,3] \) and \( B[1,4]A[1,4] = (AB)[1,4] \), and this results are not related to P-SVD. To our knowledge, there is no article discussing these in the literature.

In this note we will need the following two lemmas.

**Lemma 1.1** (see \([11,12]\)). Let \( A \in \mathbb{C}^{m \times n} \), \( B \in \mathbb{C}^{m \times k} \), \( X \in \mathbb{C}^{k \times l} \), \( C \in \mathbb{C}^{l \times n} \) and \( D \in \mathbb{C}^{l \times k} \). Then

1. \( r(A,B) = r(A) + r(B) - \dim \mathcal{R}(A) \cap \mathcal{R}(B) \);  
2. \( r(BX) = r(X) - \dim \mathcal{N}(B) \cap \mathcal{R}(X) \);  
3. \( r\begin{pmatrix} C \\ A \end{pmatrix} = r(A) + r\begin{pmatrix} C(I - A^\dagger A) \end{pmatrix} \);  
4. \( \max_X r(A - BXC) = \min \left\{ r[A, B], r\begin{pmatrix} A \\ C \end{pmatrix} \right\} \);  
5. \( \max_{A^{(1,3)}} r\left(D - CA^{(1,3)}B\right) = \min \left\{ r\begin{pmatrix} A^*A & A^*B \\ C & D \end{pmatrix} - r(A), r\begin{pmatrix} B \\ D \end{pmatrix} \right\} \);  
6. \( \min_{A^{(1,3)}} r\left(D - CA^{(1,3)}B\right) = r\begin{pmatrix} A^*A & A^*B \\ C & D \end{pmatrix} + r\begin{pmatrix} B \\ D \end{pmatrix} - r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \).
Lemma 1.2 (see [13]). Let \( A_{ij} \in \mathbb{C}^{m \times n_i} \) \((1 \leq i, j \leq 3)\) be given; \( X \in \mathbb{C}^{m_1 \times n_3} \) and \( Y \in \mathbb{C}^{m_3 \times n_1} \) are two arbitrary matrices. Then

\[
\min_{X,Y} r \begin{pmatrix} A_{11} & A_{12} & X \\ A_{21} & A_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{pmatrix} = r(A_{21}, A_{22}, A_{23}) + r(A_{12}) + \max \left\{ r \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - r \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}, r \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - r(A_{22}, A_{23}) \right\}.
\]

(1.10)

2. Main Results

In this section, we first give the minimal rank of \( D - B^{(1,3)} A^{(1,3)} \) with respect to any \( B^{(1,3)} \) and \( A^{(1,3)} \). Secondly, the necessary and sufficient conditions for the inclusion \( B[1,3] A[1,3] \supseteq (AB)[1,3] \) are obtained by using our previous result. Finally, we also give the necessary and sufficient conditions for \( B[1,3] A[1,3] = (AB)[1,3] \), \( B[1,4] A[1,4] \supseteq (AB)[1,4] \), and \( B[1,4] A[1,4] = (AB)[1,4] \).

Lemma 2.1. Let \( A \in \mathbb{C}^{m \times n} \), \( B \in \mathbb{C}^{n \times k} \) and \( D \in \mathbb{C}^{k \times m} \). Then

\[
\min_{B^{(1,3)},A^{(1,3)}} r(D - B^{(1,3)} A^{(1,3)}) = r \left( B^* BD + B^* A^* A \right) - \min \left\{ r \begin{pmatrix} B^* \\ A \end{pmatrix}, r \begin{pmatrix} B D \\ A^* \end{pmatrix} - r \begin{pmatrix} D \\ A^* \end{pmatrix} + n \right\}.
\]

(2.1)

Proof. The expression of \( (1,3) \)-inverses of \( A \) can be written as \( A^{(1,3)} = A^\dagger + F_A V \), where \( F_A = I - A^\dagger A \) and the matrix \( V \) is arbitrary; see [1]. By combining this fact with elementary block matrix operations, it follows that

\[
r \left( D - B^{(1,3)} A^{(1,3)} \right) = r \left( \begin{pmatrix} B^\dagger + F_B \tilde{V} \end{pmatrix} \left( A^\dagger + F_A V \right) - D \right)
\]

\[
= r \begin{pmatrix} 0 & 0 & 0 & 0 & I_n & V \\ 0 & 0 & -I_m & 0 & 0 & I_m \\ 0 & 0 & 0 & I_n & F_A & 0 \\ -B^\dagger & F_B & -D & 0 & 0 & 0 \\ I_n & 0 & A^\dagger & I_n & 0 & 0 \\ \tilde{V} & I_k & 0 & 0 & 0 & 0 \end{pmatrix} - k - m - 3n.
\]

(2.2)
Applying (1.10) to (2.2) gives

\[
\min_{B(1,3), A(1,3)} r\left(D - B(1,3) A(1,3)\right) = r\left(F_B, B^\dagger A^\dagger - D, -B^\dagger F_A\right) \\
+ \max \left\{-r\left(F_B, B^\dagger F_A\right), r\left(-D_0 A^\dagger, F_A\right) - r\left(F_B, -D_0 A^\dagger, -F_A\right)\right\}.
\]

(2.3)

By using the elementary block matrix operations, the rank of the first partitioned matrix in the right-hand side of (2.3) is simplified as follows:

\[
r\left(F_B, B^\dagger A^\dagger - D, -B^\dagger F_A\right) \\
= r \left(\begin{array}{cccccc}
-B^\dagger & F_B & -D & 0 \\
I_n & 0 & A^\dagger & -F_A
\end{array}\right) - n \\
= r \left(\begin{array}{cccccc}
B^\dagger & 0 & 0 & 0 & 0 & 0 \\
B^\dagger & -B^\dagger & I_k & -B^\dagger B & -D & 0 & 0 \\
0 & I_n & 0 & A^\dagger & -I_n + A^\dagger A & A^\dagger \\
0 & 0 & 0 & 0 & 0 & A^\dagger
\end{array}\right) - n - r\left(A^\dagger\right) - r\left(B^\dagger\right) \\
= r \left(\begin{array}{cccc}
B^\dagger B \quad B^\dagger B \quad 0 & 0 & 0 \\
B^\dagger & 0 & I_k & -D & 0 & 0 \\
0 & I_n & 0 & -I_n & A^\dagger \\
0 & 0 & 0 & -A^\dagger & -A^\dagger A & A^\dagger
\end{array}\right) - n - r\left(A\right) - r\left(B\right) \\
= r \left(\begin{array}{cc}
B^\dagger B D & B^\dagger \\
A^\dagger & A^\dagger A
\end{array}\right) + k - r\left(A\right) - r\left(B\right).
\]

(2.4)

Using the formula \(r(AB) \leq \min\{r(A), r(B)\}\) together with the fact that

\[
\left(\begin{array}{cc}
B^* B & 0 \\
0 & A^* A
\end{array}\right) \left(\begin{array}{cc}
B^\dagger B D & B^\dagger \\
A^\dagger & A^\dagger A
\end{array}\right) = \left(\begin{array}{cc}
B^* B D & B^* \\
A^* & A^* A
\end{array}\right),
\]

\[
\left(\begin{array}{cc}
B^\dagger B^\dagger & 0 \\
0 & A^\dagger \left(A^\dagger\right)^*\end{array}\right) \left(\begin{array}{cc}
B^* B D & B^* \\
A^* & A^* A
\end{array}\right) = \left(\begin{array}{cc}
B^\dagger B D & B^\dagger \\
A^* & A^* A
\end{array}\right)
\]

(2.5)

means that

\[
r\left(\begin{array}{cc}
B^\dagger B D & B^\dagger \\
A^\dagger & A^\dagger A
\end{array}\right) = r\left(\begin{array}{cc}
B^* B D & B^* \\
A^* & A^* A
\end{array}\right).
\]

(2.6)
Substituting (2.6) into (2.4) yields

\[
\begin{aligned}
    r(F_B, B^\dagger A^\dagger - D, -B^\dagger F_A) &= r\left(\begin{bmatrix} B^* BD \\ A^* & B^* \\ A^* A \end{bmatrix} \right) + k - r(A) - r(B).
\end{aligned}
\]

Similarly, we obtain

\[
\begin{aligned}
    r(F_B, B^\dagger F_A) &= r\left(\begin{bmatrix} B^* \\ A \end{bmatrix} \right) + k - r(A) - r(B), \\
    r\left(\begin{bmatrix} -D & 0 \\ A^\dagger & -F_A \end{bmatrix} \right) &= r\left(\begin{bmatrix} A^* \\ D \end{bmatrix} \right) + n - r(A), \\
    r\left(\begin{bmatrix} F_B & -D & 0 \\ 0 & A^\dagger & -F_A \end{bmatrix} \right) &= r\left(\begin{bmatrix} BD \\ A^* \end{bmatrix} \right) + n + k - r(A) - r(B).
\end{aligned}
\]

It is always true that \( R(I - A^\dagger A) = \mathcal{A}(A) \). Therefore,

\[
r(F_A) = r\left( I - A^\dagger A \right) = n - r(A).
\]

Substituting (2.7)–(2.9) into (2.3) yields (2.1). \( \square \)

**Theorem 2.2.** Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{n \times k} \). Then the following statements are equivalent:

1. \( B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\} \);
2. \( r(A^* AB, B) + r(A) = r(AB) + \min\{ r(A^*, B), \max\{ n + r(A) - m, n + r(B) - k \} \} \).

**Proof.** We know that \( B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\} \) is equivalent to saying that for an arbitrary \( (AB)\{1,3\}^{-1} \)-inverse \((AB)^{(1,3)}\), there are \( (1,3) \)-inverses \( A^{(1,3)} \) and \( B^{(1,3)} \) satisfying \( B^{(1,3)} A^{(1,3)} = (AB)^{(1,3)} \). That is,

\[
B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\} \iff \max_{(AB)^{(1,3)}, A^{(1,3)}} \min_{B^{(1,3)}} r\left( (AB)^{(1,3)} - B^{(1,3)} A^{(1,3)} \right) = 0.
\]

By using the formula (2.1), we get

\[
\begin{aligned}
    \min_{B^{(1,3)}, A^{(1,3)}} r\left( (AB)^{(1,3)} - B^{(1,3)} A^{(1,3)} \right) &= r\left( \begin{bmatrix} B^* B (AB)^{(1,3)} \\ A^* \\ A^* A \end{bmatrix} \right) - \min\left( r\left( \begin{bmatrix} B^* \\ A^* \\ A^* A \end{bmatrix} \right), r\left( B (AB)^{(1,3)} \right) - r\left( \begin{bmatrix} (AB)^{(1,3)} \\ A^* \end{bmatrix} \right) + n \right).
\end{aligned}
\]
Using the formulas (1.9) and (1.8) together with elementary block matrix operations, the maximal and minimal ranks of first partitioned matrix in the right-hand side of (2.11) are as follows:

\[
\begin{align*}
\min_{(AB)^{(1,3)}} \ r \left( \begin{array}{cc} B^* B (AB)^{(1,3)} & B^* \\ A^* & A^* A \end{array} \right) &= \min_{(AB)^{(1,3)}} \left[ r \left( \begin{array}{cc} 0 & B^* \\ A^* & A^* A \end{array} \right) - \left( -B^* B \right) (AB)^{(1,3)} (I, 0) \right] \\
&= r \left( \begin{array}{ccc} B^* A^* A & B^* A^* & 0 \\ -B^* B & 0 & B^* \\ 0 & A^* A & A^* A \end{array} \right) + r \left( \begin{array}{cc} I & 0 \\ 0 & B^* \\ A^* A & A^* A \end{array} \right) - r \left( \begin{array}{cc} AB & 0 & 0 \\ 0 & I & 0 \\ -B^* B & 0 & B^* \\ 0 & A^* A & A^* A \end{array} \right) \\
&= r \left( \begin{array}{c} B^* A^* A \\ B^* \end{array} \right) + r(A) - r(AB) = \max_{(AB)^{(1,3)}} r \left( \begin{array}{cc} B^* B (AB)^{(1,3)} & B^* \\ A^* & A^* A \end{array} \right).
\end{align*}
\]

Therefore, for an arbitrary \([1, 3]\)-inverse \((AB)^{(1,3)}\),

\[
\begin{align*}
 r \left( \begin{array}{cc} B^* B (AB)^{(1,3)} & B^* \\ A^* & A^* A \end{array} \right) &= r \left( \begin{array}{c} B^* A^* A \\ B^* \end{array} \right) + r(A) - r(AB). 
\end{align*}
\]

Using formulas (1.6) and (1.5), we get

\[
\begin{align*}
 r \left( \begin{array}{c} (AB)^{(1,3)} \\ A^* \end{array} \right) - r \left( \begin{array}{c} (AB)^{(1,3)} \\ A^* \end{array} \right) &= r \left[ B (AB)^{(1,3)} (I - AA^t) \right] - r \left[ (AB)^{(1,3)} (I - AA^t) \right] \\
&= - \dim \mathcal{N}(B) \cap \mathcal{R} \left[ (AB)^{(1,3)} (I - AA^t) \right].
\end{align*}
\]

Substituting (2.13) and (2.14) into (2.11) produces

\[
\begin{align*}
\min_{(AB)^{(1,3)}, (A)^{(1,3)}} r \left[ (AB)^{(1,3)} - B^{(1,3)} A^{(1,3)} \right] &= r \left( \begin{array}{c} B^* A^* A \\ B^* \end{array} \right) + r(A) - r(AB) \\
&\quad - \min \left\{ r \left( \begin{array}{c} B^* \\ A \end{array} \right), n - \dim \mathcal{N}(B) \cap \mathcal{R} \left[ (AB)^{(1,3)} (I - AA^t) \right] \right\}. 
\end{align*}
\]
Furthermore, we have

\[
\max_{(AB)^{(1,3)} B^{(1,3)}, A^{(1,3)}} \min r \left[ (AB)^{(1,3)} - B^{(1,3)} A^{(1,3)} \right] = r(B^* A^* A) + r(A) - r(AB) - \min \left\{ r(B^*), n - a \right\},
\]

where \( a = \max_{(AB)^{(1,3)}} \dim \mathcal{A}(B) \cap \mathcal{R}[(AB)^{(1,3)} (I - AA^\dagger)]. \)

Next, we want to prove that \( a \) is equal to \( \min\{k - r(B), \ m - r(A)\}. \) First observe that \( a \leq \min\{k - r(B), \ m - r(A)\} \) since \( a \leq \dim \mathcal{A}(B) = k - r(B) \) and \( a \leq \max_{(AB)^{(1,3)}} r[(AB)^{(1,3)} (I - AA^\dagger)] \leq r(I - AA^\dagger) = \dim \mathcal{A}(A^*). \) Therefore, \( a = \min\{k - r(B), \ m - r(A)\} \) holds if and only if there is a \( \{1, 3\}\)-inverse \((AB)^{(1,3)}\) such that

\[
\dim \mathcal{A}(B) \cap \mathcal{R}[(AB)^{(1,3)} (I - AA^\dagger)] = \min\{k - r(B), m - r(A)\}.
\]

Suppose that \( m - r(A) \leq k - r(B). \) Also note that \( r[(AB)^{(1,3)} (I - AA^\dagger)] \leq m - r(A) \) for arbitrary \( \{1, 3\}\)-inverses \((AB)^{(1,3)}\). Therefore, for some \((AB)^{(1,3)}\), (2.17) holds if and only if there is a \( \{1, 3\}\)-inverse \((AB)^{(1,3)}\) such that \( \mathcal{R}[(AB)^{(1,3)} (I - AA^\dagger)] \subseteq \mathcal{A}(B) \) and \( r[(AB)^{(1,3)} (I - AA^\dagger)] = m - r(A) \) hold—that is,

\[
\min_{(AB)^{(1,3)}} r \left[ \begin{pmatrix} B \\ I \end{pmatrix} (AB)^{(1,3)} (I - AA^\dagger) - \begin{pmatrix} 0 \\ C \end{pmatrix} \right] = 0,
\]

where \( C \) is any \( k \times m \) matrix and \( r(C) = m - r(A). \) It follows from the formula (1.7) that \( \max_X r(I - B^\dagger B)X(I - AA^\dagger) = \min\{r(I - B^\dagger B), \ r(I - AA^\dagger)\} = m - r(A) \). Therefore, there is a matrix \( X_0 \) satisfying \( r(I - B^\dagger B)X_0(I - AA^\dagger) = m - r(A) \). Let \( C = (I - B^\dagger B)X_0(I - AA^\dagger). \) It is always true that \( r(C) = m - r(A), \ BC = 0, \) and \( B^* A^* (I - AA^\dagger) = 0. \) Use these equations together with the formula (1.9) to conclude that (2.18) holds. Therefore, if \( m - r(A) \leq k - r(B) \), then there is a \( \{1, 3\}\)-inverse \((AB)^{(1,3)}\) such that (2.17) holds.

On the other hand, suppose that \( m - r(A) > k - r(B). \) Also note that \( \dim \mathcal{A}(B) = k - r(B). \) Therefore, for some \((AB)^{(1,3)}\) (2.17) holds if and only if there is a \( \{1, 3\}\)-inverse \((AB)^{(1,3)}\) such that \( \mathcal{A}(B) = \mathcal{R}(I - B^\dagger B) \subseteq \mathcal{R}[(AB)^{(1,3)} (I - AA^\dagger)] \) holds, that is,

\[
\min_{(AB)^{(1,3)}} r \left[ I - B^\dagger B - (AB)^{(1,3)} (I - AA^\dagger) \right] X = 0,
\]

(2.19)
where $X$ is some $m \times k$ matrix. Use the formula (1.9) to find that

$$
\min_{(AB)^{(1,3)}} r \left[ I - B^t B - (AB)^{(1,3)} (I - AA^t) X \right]
= r \left( B^* A^* AB \right) - r \left( (I - AA^t) X \right)
= r \left( (I - AA^t) X \right) - r \left( (I - AA^t) X \right)
= r \left( X^* (I - AA^t) , I - B^t B \right) - r \left( X^* (I - AA^t) \right).
$$

(2.20)

We know from (2.20) that (2.19) holds if and only if there is an $m \times k$ matrix $X$ such that $\mathcal{R}(I - B^t B) \subseteq \mathcal{R}(X^*(I - AA^t))$. In fact, note that $r(I - B^t B) = \dim \mathcal{N}(B) = k - r(B)$ and $r(I - A^t A) = \dim \mathcal{N}(A^*) = m - r(A)$, and let $P_1, P_2, Q_1,$ and $Q_2$ be nonsingular matrices such that $I - B^t B = P_1 \begin{pmatrix} k-r(B) & 0 \\ 0 & 0 \end{pmatrix} Q_1$ and $I - A^t A = P_2 \begin{pmatrix} k-r(A) & 0 \\ 0 & 0 \end{pmatrix} Q_2$. Using this together with $m - r(A) > k - r(B)$ means that if $X^* = P_1 P_2^{-1}$, then $\mathcal{R}(I - B^t B) \subseteq \mathcal{R}(X^*(I - AA^t))$. Therefore, if $m - r(A) > k - r(B)$, then there is a $\{1,3\}$-inverse $(AB)^{(1,3)}$ such that (2.17) holds.

In summary, there is a $\{1,3\}$-inverse $(AB)^{(1,3)}$ such that (2.17) holds. That is, $a = \min\{k - r(B), m - r(A)\}$. Apply this to (2.16) to obtain that

$$
\max_{(AB)^{(1,3)}} \min_{B^{(1,3)}, A^{(1,3)}} r \left[ (AB)^{(1,3)} - B^{(1,3)} A^{(1,3)} \right] = r(A^* AB, B) + r(A) - r(AB)
- \min\{r(A^*, B), \max\{n + r(B) - k, n + r(A) - m\}\}
$$

(2.21)

Noting that (2.10) and letting the right-hand side in (2.21) be equal to zero, then the equivalence between (1) and (2) follows immediately.

It is obvious that $B\{1,3\}A\{1,3\} = (AB)\{1,3\}$ if and only if $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\}$ and $B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}$. Also note Theorem 2.2 and formula (1.2). It is easy to obtain the following theorem.

**Theorem 2.3.** Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times k}$. Then the following statements are equivalent:

1. $B\{1,3\}A\{1,3\} = (AB)\{1,3\}$;
2. $r(B, A^* AB) = r(B)$ and $r(A) + r(B) = r(AB) + \min\{r(A^*, B), \max\{n + r(B) - k, n + r(A) - m\}\}$.

The following theorems can be obtained by applying Theorem 2.2 or Theorem 2.3 to the product $B^* A^*$ and using the fact that $X \in D\{1,3\}$ if and only if $X^* \in D^*\{1,4\}$.
Theorem 2.4. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times k}$. Then the following statements are equivalent:

2. $r(BB^*A^*, A^*) + r(B) + r(AB) = r(AB) + \min \{r(A^*, B), \max \{n + r(A) - m, n + r(B) - k\}\}$.

Theorem 2.5. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times k}$. Then the following statements are equivalent:

2. $r(BB^*A^*, A^*) = r(A) + r(B) = r(AB) + \min \{r(A^*, B), \max \{n + r(A) - m, n + r(B) - k\}\}$.

3. Examples

In this section, we give two examples. The first example comes from [14], and they verify that $B[1,2,3]A[1,2,3] \subseteq (AB)[1,2,3]$. However, this example does not only satisfy this result. In Example 3.1, we know that this example satisfies Theorems 2.3 and 2.5, and so we have $B[1,3]A[1,3] = (AB)[1,3]$ and $B[1,4]A[1,4] = (AB)[1,4]$. In this example, we will verify these results. Secondly, we give another example which only satisfies $B[1,3]A[1,3] \supset (AB)[1,3]$ and $B[1,4]A[1,4] \supset (AB)[1,4]$.

Example 3.1. Let

\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}. \tag{3.1}
\]

It is easy to obtain that

\[
r(B, A^*AB) = r(A^*, BB^*A^*) = r(B) = r(A) = r(B, A^*) = 2. \tag{3.2}
\]

From Theorems 2.3 and 2.5, we can conclude that

\[
B[1,3]A[1,3] = (AB)[1,3], \quad B[1,4]A[1,4] = (AB)[1,4]. \tag{3.3}
\]

Now we verify this statement. Since

\[
A[1,3] = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ -a_1 - a_2 + \frac{1}{2} - a_3 + \frac{1}{2} \end{pmatrix} \mid a_1, a_2, a_3 \in \mathbb{C} \right\},
\]

\[
B[1,3] = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ a_4 & a_5 & a_6 \end{pmatrix} \mid a_4, a_5, a_6 \in \mathbb{C} \right\},
\]
we easily find that

\[
B[1,3]A[1,3] = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ a & b & c \end{pmatrix} \mid a_i \in \mathbb{C}, \ i = 1,2,\ldots,6 \right\},
\]

where \( a = a_4 + a_1 a_5 - a_1 a_6, b = a_2 a_5 - a_2 a_6 + (1/2)a_6, \) and \( c = a_3 a_5 - a_3 a_6 + (1/2)a_6. \) It is obvious that \( B[1,3]A[1,3] \subseteq (AB)[1,3]. \) If \( a_1 = a_2 = 0, \ a_3 = 1, \ a_4 = a_7, \ a_5 = a_8 + a_9, \) and \( a_6 = 2a_8, \) then we have \( a = a_7, \ b = a_8, \) and \( c = a_9, \) that is, \( B[1,3]A[1,3] \supseteq (AB)[1,3]. \) Therefore, \( B[1,3]A[1,3] = (AB)[1,3]. \)

On the other hand, since

\[
A[1,4] = \left\{ \begin{pmatrix} 1 & a_1 & -a_1 \\ 0 & a_2 & -a_2 + \frac{1}{2} \\ 0 & -a_3 + \frac{1}{2} & a_3 \end{pmatrix} \mid a_1, a_2, a_3 \in \mathbb{C} \right\},
\]

\[
B[1,4] = \left\{ \begin{pmatrix} 1 & a_4 & -a_4 \\ -1 & a_5 & 1 - a_5 \\ 0 & a_6 & -a_6 \end{pmatrix} \mid a_4, a_5, a_6 \in \mathbb{C} \right\},
\]

\[
(AB)[1,4] = \left\{ \begin{pmatrix} 1 & a_7 & -a_7 \\ -1 & a_8 & -a_8 + \frac{1}{2} \\ 0 & a_9 & -a_9 \end{pmatrix} \mid a_7, a_8, a_9 \in \mathbb{C} \right\},
\]

we easily see that

\[
B[1,4]A[1,4] = \left\{ \begin{pmatrix} 1 & d & -d \\ -1 & e & -e + \frac{1}{2} \\ 0 & f & -f \end{pmatrix} \mid a_i \in \mathbb{C}, \ i = 1,2,\ldots,6 \right\},
\]

where \( d = a_1 - (1/2)a_4 + a_2 a_4 + a_3 a_4, \ e = (1/2) - a_1 - a_3 - (1/2)a_5 + a_2 a_5 + a_3 a_5, \) and \( f = a_2 a_6 + a_3 a_6 - (1/2)a_6. \) It is obvious that \( B[1,4]A[1,4] \subseteq (AB)[1,4]. \) If \( a_1 = a_7, \ a_2 = a_7 + a_8 + a_9, \ a_3 = 1/2 - a_7 - a_8, \ a_4 = a_5 = 0 \) and \( a_6 = 1, \) then we have \( d = a_7, \ e = a_8, \) and \( f = a_9, \) that is, \( B[1,4]A[1,4] \supseteq (AB)[1,4]. \) Therefore, \( B[1,4]A[1,4] = (AB)[1,4]. \)
Example 3.2. Let

\[
A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (3.8)

It is easy to obtain that

\[
\begin{align*}
& r(A) = r(B) = r(AB) = 2, \\
& r(B, A^*AB) = r(A^*, BB^*A^*) = r(B, A^*) = 3.
\end{align*}
\] (3.9)

From Theorems 2.2 and 2.4, we can find that

\[
B\{1, 3\} A\{1, 3\} \supseteq (AB)\{1, 3\}, \quad B\{1, 4\} A\{1, 4\} \supseteq (AB)\{1, 4\}.
\] (3.10)

Furthermore, note that \(r(B, A^*AB) = r(A^*, BB^*A^*) = 3 \neq r(B) = r(A) = 2\). Using Theorems 2.3 and 2.5, we can conclude that

\[
B\{1, 3\} A\{1, 3\} \supset (AB)\{1, 3\}, \quad B\{1, 4\} A\{1, 4\} \supset (AB)\{1, 4\}.
\] (3.11)

Now we verify this statement. Since

\[
A\{1, 3\} = \begin{cases} 
\begin{pmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ -a_1 -a_2 + \frac{1}{2} -a_3 + \frac{1}{2} \\ a_4 & a_5 & a_6 \end{pmatrix} & | a_1, a_2, \ldots, a_6 \in \mathbb{C} 
\end{cases},
\]

\[
B\{1, 3\} = \begin{cases} 
\begin{pmatrix} 2 & 1 & 1 \\ \frac{3}{3} & \frac{3}{3} & \frac{1}{3} \\ -1 & 1 & 0 \\ \frac{3}{3} & \frac{3}{3} & 0 \end{pmatrix} & | a_7, a_8, a_9, a_{10} \in \mathbb{C} 
\end{cases},
\] (3.12)

\[
(AB)\{1, 3\} = \begin{cases} 
\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ a_{11} & a_{12} & a_{13} \end{pmatrix} & | a_{11}, a_{12}, a_{13} \in \mathbb{C} 
\end{cases}.
\]
we easily get that

\[
B_{1,3}A_{1,3} = \begin{pmatrix}
\frac{2}{3} + \frac{2}{3}a_1 & -1 + \frac{2}{3}a_2 & -1 + \frac{2}{3}a_3 \\
-\frac{1}{3} - \frac{1}{3}a_1 & \frac{1}{3} - \frac{1}{3}a_2 & \frac{1}{3} - \frac{1}{3}a_3 \\
a & b & c
\end{pmatrix}
\left| a_1, a_2, \ldots, a_{10} \in \mathbb{C} \right.,
\]

(3.13)

where \( a = a_7 + a_1a_8 - a_1a_9 + a_4a_{10}, b = (1/2)a_9 + a_2a_8 - a_2a_9 + a_5a_{10}, \) and \( c = (1/2)a_9 + a_3a_8 - a_3a_9 + a_6a_{10}. \) It is obvious that if \( a_1 = 1/2, a_2 = 1/4, a_3 = 1/4, a_4 = a_6 = a_9 = 0, a_5 = a_{12} = a_{13}, a_7 = 2a_{13} + a_{11}, a_9 = 4a_{13}, \) and \( a_{10} = 1, \) then

\[
\begin{pmatrix}
\frac{2}{3} + \frac{2}{3}a_1 & -1 + \frac{2}{3}a_2 & -1 + \frac{2}{3}a_3 \\
-\frac{1}{3} - \frac{1}{3}a_1 & \frac{1}{3} - \frac{1}{3}a_2 & \frac{1}{3} - \frac{1}{3}a_3 \\
a & b & c
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
-\frac{1}{2} & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}.
\]

(3.14)

That is, \( B_{1,3}A_{1,3} \supseteq (AB)_{1,3}. \) Furthermore, note that if \( a_1 \neq 1/2, \) then there are some \( B_{1,3}A_{1,3} \) which do not belong to \( (AB)_{1,3}. \) Therefore, \( B_{1,3}A_{1,3} \supset (AB)_{1,3}. \)

On the other hand, because

\[
A_{1,4} = \left\{ \begin{pmatrix}
1 & a_1 & -a_1 \\
0 & a_2 & -a_2 + \frac{1}{2} \\
0 & a_3 & -a_3 + \frac{1}{2} \\
0 & a_4 & -a_4
\end{pmatrix} \left| a_1, a_2, a_3, a_4 \in \mathbb{C} \right. \right\},
\]

(3.15)

\[
B_{1,4} = \left\{ \begin{pmatrix}
a_5 & -a_5 + 1 & a_5 - 1 & a_6 \\
a_7 & -a_7 & a_7 + 1 & a_8 \\
a_9 & -a_9 & a_9 & a_{10}
\end{pmatrix} \left| a_5, a_6, \ldots, a_{10} \in \mathbb{C} \right. \right\},
\]

\[
(AB)_{1,4} = \left\{ \begin{pmatrix}
1 & a_{11} & -a_{11} \\
-\frac{1}{2} & a_{12} & -a_{12} + \frac{1}{2} \\
0 & a_{13} & -a_{13}
\end{pmatrix} \left| a_{11}, a_{12}, a_{13} \in \mathbb{C} \right. \right\},
\]

we easily obtain that

\[
B_{1,4}A_{1,4} = \left\{ \begin{pmatrix}
a_5 & d & -d \\
a_7 & e & -e + \frac{1}{2} \\
a_9 & f & -f
\end{pmatrix} \left| a_1, a_2, \ldots, a_{10} \in \mathbb{C} \right. \right\},
\]

(3.16)
where $d = a_2 - a_3 + a_1a_5 - a_2a_5 + a_3a_5 + a_4a_6$, $e = a_3 + a_1a_7 - a_2a_7 + a_4a_7$, and $f = a_1a_9 - a_2a_9 + a_3a_9 + a_4a_{10}$. It is obvious that if $a_1 = a_{11}$, $a_2 = a_5 = a_8 = a_9 = 0$, $a_3 = a_{11} + 2a_{12}$, $a_4 = a_{13}$, $a_5 = a_{10} = 1$ and $a_7 = -1/2$, then

$$
\begin{pmatrix}
    a_5 & d & -d \\
    a_7 & e & -e + \frac{1}{2} \\
    a_9 & f & -f
\end{pmatrix}
= \begin{pmatrix}
    1 & a_{11} & -a_{11} \\
    -\frac{1}{2} & a_{12} & -a_{12} + \frac{1}{2} \\
    0 & -a_{13} & -a_{13}
\end{pmatrix}.
$$

That is, $B\{1,4\}A\{1,4\} \supseteq (AB)\{1,4\}$. Furthermore, note that if $a_5 \neq 1$, then there are some $B^{(1,4)}A^{(1,4)}$ which do not belong to $(AB)\{1,4\}$. Therefore, $B\{1,4\}A\{1,4\} \supset (AB)\{1,4\}$.

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**References**


