Research Article

Hyers-Ulam Stability of Differential Equation

\[ y'' + 2xy' - 2ny = 0 \]

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We solve the inhomogeneous differential equation of the form

\[ y'' + 2xy' - 2ny = \sum_{m=0}^{\infty} a_n x^m, \]

where \( n \) is a nonnegative integer, and apply this result to the proof of a local Hyers-Ulam stability of the differential equation \( y'' + 2xy' - 2ny = 0 \) in a special class of analytic functions.

1. Introduction

Assume that \( X \) and \( Y \) are a topological vector space and a normed space, respectively, and that \( I \) is an open subset of \( X \). If for any function \( f : I \to Y \) satisfying the differential inequality

\[ \| a_n(x) y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) + h(x) \| \leq \varepsilon \quad (1.1) \]

for all \( x \in I \) and for some \( \varepsilon \geq 0 \), there exists a solution \( f_0 : I \to Y \) of the differential equation

\[ a_n(x) y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) + h(x) = 0 \quad (1.2) \]

such that \( \| f(x) - f_0(x) \| \leq K(\varepsilon) \) for any \( x \in I \), where \( K(\varepsilon) \) is an expression of \( \varepsilon \) only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain \( I \) is not the whole space \( X \)). We may apply this terminology for other differential equations. For more detailed definition of the Hyers-Ulam stability, refer to [1–6].

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [7, 8]). Here, we will introduce a result of Alsina and Ger (see [9]): If a differentiable function \( f : I \to \mathbb{R} \) is a solution of the differential inequality
The general solution of the inhomogeneous differential equation $y'(x) = y(x)$ is given by

$$|y'(x) - y(x)| \leq \varepsilon,$$

where $I$ is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \leq 3\varepsilon$ for any $x \in I$.

This result of Alsina and Ger has been generalized by Takahasi et al.: They proved in [10] that the Hyers-Ulam stability holds true for the Banach space-valued differential equation $y'(x) = Ay(x)$ (see also [11]).

Using the conventional power series method, the author in [12] investigated the general solution of the inhomogeneous Legendre differential equation of the form

$$\left(1 - x^2\right)y''(x) - 2xy'(x) + p(p + 1)y(x) = \sum_{m=0}^{\infty} a_m x^m$$

(1.3)

under some specific conditions, where $p$ is a real number and the convergence radius of the power series is positive. Moreover, he applied this result to prove that every analytic function is expressed by power series is positive. Moreover, using the idea from [13–15], we will investigate the general solution of the inhomogeneous differential equation:

$$\left(1 - x^2\right)y''(x) - 2xy'(x) + p(p + 1)y(x) = \sum_{m=0}^{\infty} a_m x^m,$$

(1.8)

where the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is $\rho > 0$, whose value is in general permitted to have infinity. Moreover, using the idea from [12–14], we will prove the Hyers-Ulam stability of the differential equation (1.6) in a class of special analytic functions (see the class $C_K$ in Section 3).

In this paper, $\mathbb{N}_0$ denotes the set of all nonnegative integers.
2. General Solution of (1.8)

In the following theorem, we solve the inhomogeneous differential equation (1.8).

**Theorem 2.1.** Assume that $n$ is a nonnegative integer, the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is $\rho > 0$, and that there exists a real number $\mu \geq 0$ with

$$|a_{2m}| \leq \mu^2 m(2m + 2)(2m + 1)|a_{2m+2}| \quad (\text{if } n \text{ is odd}),$$

$$|a_{2m+1}| \leq \mu^2 m(2m + 3)(2m + 2)|\beta_{2m+3}| \quad (\text{if } n \text{ is even})$$

(2.1)

for all sufficiently large integers $m$, where

$$a_{2m} = \frac{2^{m-1}}{(2m)!} \sum_{k=0}^{m-2} \frac{(2k)!}{2^k} a_{2k} \prod_{i=k+1}^{m-1} (n - 2i),$$

$$\beta_{2m+1} = \frac{2^{m-1}}{(2m + 1)!} \sum_{k=0}^{m-2} \frac{(2k + 1)!}{2^k} a_{2k+1} \prod_{i=k+1}^{m-1} [n - (2i + 1)]$$

(2.2)

for any $m \in \{2, 3, \ldots\}$. Let us define $\rho_0 = \min\{\rho, 1/\mu\}$ and $1/0 = \infty$. Every solution $y : (-\rho_0, \rho_0) \to \mathbb{C}$ of the inhomogeneous differential equation (1.8) can be represented by

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} a_{m-2} \frac{a_m}{m(m-1)} x^m + \sum_{m=2}^{\infty} a_{2m} x^{2m} + \sum_{m=2}^{\infty} \beta_{2m+1} x^{2m+1},$$

(2.3)

where $y_h(x)$ is a solution of the homogeneous differential equation (1.6).

**Proof.** Assume that a function $y : (-\rho_0, \rho_0) \to \mathbb{C}$ is given by (2.3). We first prove that the function $y_p(x)$, defined by $y(x) - y_h(x)$, satisfies the inhomogeneous differential equation (1.8). Since

$$y_p'(x) = \sum_{m=0}^{\infty} a_{m} x^m + \sum_{m=1}^{\infty} (2m + 2)(2m + 1) a_{2m+2} x^{2m+1} + \sum_{m=1}^{\infty} (2m + 3)(2m + 2) \beta_{2m+3} x^{2m+1},$$

$$y_p''(x) = \sum_{m=0}^{\infty} a_{m} x^{m} + \sum_{m=1}^{\infty} (2m + 2)(2m + 1) a_{2m+2} x^{2m+1} + \sum_{m=1}^{\infty} (2m + 3)(2m + 2) \beta_{2m+3} x^{2m+1},$$

(2.4)
we have

\[
y''_p(x) + 2xy'_p(x) - 2ny_p(x) = \sum_{m=0}^{\infty} a_m x^m + \sum_{m=1}^{\infty} (2m + 2)(2m + 1) \alpha_{2m+2} x^{2m} + \sum_{m=2}^{\infty} 2a_{m-2} x^m + \sum_{m=2}^{\infty} (2m + 3)(2m + 2) \beta_{2m+3} x^{2m+1} \\
+ \sum_{m=2}^{\infty} 4ma_{2m} x^{2m} + \sum_{m=2}^{\infty} 2(2m + 1) \beta_{2m+1} x^{2m+1} \\
- \sum_{m=2}^{\infty} \frac{2n a_{m-2}}{m(m-1)} x^m - \sum_{m=2}^{\infty} 2n \alpha_{2m} x^{2m} - \sum_{m=2}^{\infty} 2n \beta_{2m+1} x^{2m+1} \\
= \sum_{m=0}^{\infty} a_m x^m + 12 \alpha_4 x^2 + \sum_{m=2}^{\infty} (2m + 2)(2m + 1) \alpha_{2m+2} x^{2m} + 20 \beta_5 x^3 \\
+ \sum_{m=2}^{\infty} \frac{2(m-n)}{m(m-1)} a_{m-2} x^m - \sum_{m=2}^{\infty} 2(n - 2m) \alpha_{2m} x^{2m} \\
- \sum_{m=2}^{\infty} 2[n - (2m + 1)] \beta_{2m+1} x^{2m+1}
\]  

(2.5)

for all \( x \in (-\rho_0, \rho_0) \).

It is not difficult to see that

\[
(2m + 2)(2m + 1) \alpha_{2m+2} = 2(n - 2m) \alpha_{2m} + \frac{n - 2m}{m(m-1)} a_{2m-2},
\]

\[
(2m + 3)(2m + 2) \beta_{2m+3} = 2[n - (2m + 1)] \beta_{2m+1} + \frac{n - (2m + 1)}{(2m + 1)m} a_{2m-1}
\]  

(2.6)

for any \( m \in \mathbb{N} \). Hence, we obtain

\[
y''_p(x) + 2xy'_p(x) - 2ny_p(x) = \sum_{m=0}^{\infty} a_m x^m + 12 \alpha_4 x^2 + \sum_{m=2}^{\infty} \frac{n - 2m}{m(m-1)} a_{2m-2} x^{2m} + 20 \beta_5 x^3 \\
+ \sum_{m=2}^{\infty} \frac{n - (2m + 1)}{m(m+1)} a_{2m-1} x^{2m+1} + \sum_{m=2}^{\infty} \frac{2(m-n)}{m(m-1)} a_{m-2} x^m
\]  

(2.7)

which proves that \( y_p(x) \) is a particular solution of the inhomogeneous equation (1.8).
We now apply the ratio test to the power series expression of \( y_p(x) \). If \( n \) is an odd integer not less than 0, then \( \beta_{n+2} = \beta_{n+4} = \beta_{n+6} = \cdots = 0 \). Hence, the power series \( \sum_{m=1}^{\infty} \beta_{2m+1} x^{2m+1} \) is a polynomial. And it follows from the first conditions of (2.1) and (2.6) that

\[
\lim_{m \to \infty} \left| \frac{\beta_{2m+3}}{\beta_{2m+1}} \right| = \lim_{m \to \infty} \left| \frac{n - (2m + 1)}{(2m + 3)(2m + 1)} \right| \leq \frac{1}{2}
\]

If \( n \geq 0 \) is an even integer, then we have \( \alpha_{n+2} = \alpha_{n+4} = \alpha_{n+6} = \cdots = 0 \). Thus, for each even integer \( n \geq 0 \), the power series \( \sum_{m=1}^{\infty} \alpha_{2m+1} x^{2m} \) is a polynomial. By the second conditions in (2.1) and (2.6), we get

\[
\lim_{m \to \infty} \left| \frac{\beta_{2m+3}}{\beta_{2m+1}} \right| = \lim_{m \to \infty} \left| \frac{n - (2m + 1)}{(2m + 3)(2m + 1)} \right| \leq \frac{1}{2}
\]

Therefore, the power series expression of \( y_p(x) \) converges for all \( x \in (-\rho_0, \rho_0) \).

Moreover, the convergence region of the power series for \( y_p(x) \) is the same as those of power series for \( y_p'(x) \) and \( y_p''(x) \). In this paper, the convergence region will denote the maximum open set where the relevant power series converges. Hence, the power series expression for \( y_p''(x) + 2xy_p'(x) - 2ny_p(x) \) has the same convergence region as that of \( y_p(x) \). This implies that \( y_p(x) \) is well defined on \((-\rho_0, \rho_0)\) and so does for \( y(x) \) in (2.3) because \( y_p(x) \) converges for all \( x \in \mathbb{R} \) under our hypotheses.

Since every solution to (1.8) can be expressed as a sum of a solution \( y_h(x) \) of the homogeneous equation and a particular solution \( y_p(x) \) of the inhomogeneous equation, every solution of (1.8) is certainly in the form of (2.3).

\[\Box\]

Remark 2.2. We might have thought that the conditions presented in (2.1) were too strong. However, we can show that some familiar sequences \( \{a_m\} \) satisfy the conditions in (2.1). For example, let \( n = 0 \) and \( a_0 = a_1 = 1, a_{2m} = a_{2m+1} = 1/(m - 1)! \) for all \( m \in \mathbb{N} \) and choose an arbitrary \( \mu > 0 \). Then, by some manipulations, we can show that the coefficients sequence
{a_m} satisfies the second condition of (2.1) for all sufficiently large integers \(m\) as we see in the following:

\[
\mu^2 m(2m + 3)(2m + 2)|\beta_{2m+3}| = \frac{\mu^2}{(m-1)!} \left| 1 + \sum_{k=1}^{m-1} (-1)^k \right| \geq \frac{1}{(m-1)!} = |a_{2m+1}|.
\]

(2.10)

3. Hyers-Ulam Stability of (1.6)

In this section, let \(n\) be a nonnegative integer and let \(\rho\) be a constant with \(0 < \rho \leq \infty\). For a given \(K \geq 0\), let us denote by \(C_K\) the set of all functions \(y : (-\rho, \rho) \to \mathbb{C}\) with the properties (a) and (b):

(a) \(y(x)\) is represented by a power series \(\sum_{m=0}^{\infty} b_m x^m\) whose radius of convergence is at least \(\rho\);

(b) it holds true that \(\sum_{m=0}^{\infty} |a_m x^m| \leq K \sum_{m=0}^{\infty} a_m x^m\) for all \(x \in (-\rho, \rho)\), where \(a_m = (m+2)(m+1)b_{m+2} + 2(m-n)b_m\) for each \(m \in \mathbb{N}_0\).

It should be remarked that the power series \(\sum_{m=0}^{\infty} a_m x^m\) in (b) has the same radius of convergence as that of \(\sum_{m=0}^{\infty} b_m x^m\) given in (a).

In the following theorem, we prove that if an analytic function satisfies some given conditions, then it can be approximated by a combination of integrals of the error function (see the last part of Section 1 or [16, §7.2.2]).

**Theorem 3.1.** Let \(n\) be a nonnegative integer. For given constants \(K\) and \(\rho\) with \(K \geq 0\) and \(0 < \rho \leq \infty\), suppose that \(y : (-\rho, \rho) \to \mathbb{C}\) is a function which belongs to \(C_K\). Assume that there exist constants \(\mu, \nu \geq 0\) satisfying

\[
\nu^2 |a_{2m+2}| \leq |a_{2m}| \leq \mu^2 m(2m + 2)(2m + 1)|a_{2m+2}|,
\]

(3.1)

\[
\nu^2 |\beta_{2m+3}| \leq |a_{2m+1}| \leq \mu^2 m(2m + 3)(2m + 2)|\beta_{2m+3}|
\]

(3.2)

for all \(m \in \mathbb{N}\). (See the definitions of \(a_m\) and \(\beta_m\) given in Theorem 2.1. Indeed, it is sufficient for the second inequalities in (3.1) and (3.2) to hold true for all sufficiently large integers \(m\).) Let us define \(\rho_0 = \min\{|\rho|, 1/\mu\}\), where \(1/0 = \infty\). If the function \(y\) satisfies the differential inequality

\[
|y''(x) + 2xy'(x) - 2ny(x)| \leq \varepsilon
\]

(3.3)
for all \( x \in (-\rho_0, \rho_0) \) and for some \( \varepsilon \geq 0 \), then there exists a solution \( y_h : \mathbb{R} \to \mathbb{C} \) of the differential equation (1.6) such that

\[
|y(x) - y_h(x)| \leq \left( \frac{1}{2} + \frac{1}{y^2} \right) K\varepsilon x^2
\]

for any \( x \in (-\rho_0, \rho_0) \).

Proof. Since \( y \in C_K \), it follows from (a) and (b) that

\[
y''(x) + 2xy'(x) - 2ny(x) = \sum_{m=0}^{\infty} [(m+2)(m+1)b_{m+2} + 2(m-n)b_m] x^m = \sum_{m=0}^{\infty} a_m x^m
\]

for all \( x \in (-\rho, \rho) \). It follows from the last equality and (3.3) that

\[
\left| \sum_{m=0}^{\infty} a_m x^m \right| \leq \varepsilon
\]

for any \( x \in (-\rho_0, \rho_0) \). This inequality, together with (b), yields that

\[
\sum_{m=0}^{\infty} |a_m x^m| \leq K \left| \sum_{m=0}^{\infty} a_m x^m \right| \leq K\varepsilon
\]

for each \( x \in (-\rho_0, \rho_0) \).

By Abel’s formula (see [17, Theorem 6.30]), we have

\[
\sum_{m=0}^{p} \left| a_m x^m \right| \frac{1}{(m+2)(m+1)} = \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left| a_m x^m \right| - \sum_{m=0}^{p-1} \left( \sum_{i=0}^{m} |a_i x^i| \right) \left[ \frac{1}{(m+3)(m+2)} - \frac{1}{(m+2)(m+1)} \right]
\]

\[
= \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left| a_m x^m \right|
\]

\[
+ \sum_{m=0}^{p-1} \left( \sum_{i=0}^{m} |a_i x^i| \right) \frac{2}{(m+1)(m+2)(m+3)}
\]

\[
\leq \sum_{m=0}^{\infty} K\varepsilon \frac{2}{(m+1)(m+2)(m+3)}
\]

\[
= \frac{K}{2} \varepsilon
\]
for any $x \in (-\rho_0, \rho_0)$ and $p \in \mathbb{N}$, since

$$
\sum_{m=0}^{\infty} \frac{2}{(m+1)(m+2)(m+3)} = \sum_{m=0}^{\infty} \left\{ \left( \frac{1}{m+1} - \frac{1}{m+2} \right) + \left( \frac{1}{m+3} - \frac{1}{m+2} \right) \right\} = \frac{1}{2}
$$

(3.9)

Hence, it follows from (3.7) and (3.8) that

$$
\sum_{m=2}^{\infty} \left| \frac{a_{m-2}}{m(m-1)} x^m \right| = x^2 \sum_{m=0}^{\infty} \left| a_m x^m \right| \left( \frac{1}{m+2}(m+1) \right) \leq \frac{K}{2} \varepsilon x^2
$$

(3.10)

for each $x \in (-\rho_0, \rho_0)$.

Finally, it follows from Theorem 2.1, (3.1), (3.2), (3.7), and (3.10) that there exists a solution function $y_h : \mathbb{R} \to \mathbb{C}$ of the homogeneous differential equation (1.6) such that

$$
|y(x) - y_h(x)| \leq \sum_{m=2}^{\infty} \left| \frac{a_{m-2}}{m(m-1)} x^m \right| + \sum_{m=2}^{\infty} (a_{2m-2}) \left| x^{2m-2} \right| + \sum_{m=2}^{\infty} (a_{2m-1}) \left| x^{2m-1} \right|
$$

$$
\leq \frac{K}{2} \varepsilon x^2 + \frac{1}{\nu^2} \sum_{m=2}^{\infty} (a_{2m-2}) \left| x^{2m-2} \right| + \frac{1}{\nu^2} \sum_{m=2}^{\infty} (a_{2m-1}) \left| x^{2m-1} \right|
$$

(3.11)

$$
\leq \left( \frac{1}{2} + \frac{1}{\nu^2} \right) K \varepsilon x^2
$$

for all $x \in (-\rho_0, \rho_0)$. \hfill \Box

If $\rho$ is finite, then the local Hyers-Ulam stability of the differential equation (1.6) immediately follows from Theorem 3.1.

**Corollary 3.2.** Let $n$ be a nonnegative integer. For given constants $K$ and $\rho$ with $K \geq 0$ and $0 < \rho < \infty$, suppose that $y : (-\rho, \rho) \to \mathbb{C}$ is a function which belongs to $C_K$. Assume that there exist constants $\mu, \nu \geq 0$ satisfying the conditions in (3.1) and (3.2) for all $m \in \mathbb{N}$. (It is sufficient for the second inequalities in (3.1) and (3.2) to hold true for all sufficiently large integers $m$.) Let us define $\rho_0 = \min\{\rho, 1/\mu\}$ and $1/0 = \infty$. If the function $y$ satisfies the differential inequality (3.3) for all $x \in (-\rho_0, \rho_0)$ and for some $\varepsilon \geq 0$, then there exists a solution $y_h : \mathbb{R} \to \mathbb{C}$ of the differential equation (1.6) such that

$$
|y(x) - y_h(x)| \leq \left( \frac{1}{2} + \frac{1}{\nu^2} \right) K \rho^2 \varepsilon
$$

(3.12)

for any $x \in (-\rho_0, \rho_0)$.

We now deal with an asymptotic behavior of functions in $C_K$ under the additional conditions (3.1) and (3.2).

**Corollary 3.3.** Let $n$ be a nonnegative integer. For given constants $K$, $\rho$, and $\rho_1$ with $K \geq 0$ and $0 < \rho_1 < \rho \leq \infty$, suppose that $y : (-\rho, \rho) \to \mathbb{C}$ is a function belonging to $C_K$. Assume that there exist constants $\mu \geq 0$ and $\nu > 0$ satisfying the conditions in (3.1) and (3.2) for any $m \in \mathbb{N}$. (It is sufficient
for the second inequalities in (3.1) and (3.2) to hold true for all sufficiently large integers \( m \). Then there exists a solution \( y_h : \mathbb{R} \to \mathbb{C} \) of the differential equation (1.6) such that

\[
|y(x) - y_h(x)| = O(x^2)
\]

as \( x \to 0 \).

Proof. Since \( y \in C_K \), it follows from the first 4 lines of the proof of Theorem 3.1 that

\[
y''(x) + 2xy'(x) - 2ny(x) = \sum_{m=0}^{\infty} a_m x^m
\]

for all \( x \in (-\rho, \rho) \). As was remarked in the first part of Section 3, the radius of convergence of the power series \( \sum_{m=0}^{\infty} a_m x^m \) is same as that of \( \sum_{m=0}^{\infty} b_m x^m (= y(x)) \), that is, it is at least \( \rho \). Since \( 0 < \rho_1 < \rho \), if we set \( \rho_0 = \min\{\rho_1, 1/\mu\} \), then there exists a constant \( \delta > 0 \) such that

\[
\left| y''(x) + 2xy'(x) - 2ny(x) \right| = \left| \sum_{m=0}^{\infty} a_m x^m \right| \leq \delta
\]

for any \( x \in (-\rho_0, \rho_0) \).

According to Theorem 3.1, there exists a solution \( y_h : \mathbb{R} \to \mathbb{C} \) of the differential equation (1.6) satisfying

\[
|y(x) - y_h(x)| \leq \left( \frac{1}{2} + \frac{1}{\nu^2} \right) K\delta x^2
\]

for any \( x \in (-\rho_0, \rho_0) \). Hence, we have

\[
|y(x) - y_h(x)| = O(x^2)
\]

as \( x \to 0 \). \( \Box \)

4. An Example

The conditions in (3.1) and (3.2) may seem too strong to construct some examples for the coefficients \( a_m \)'s. In this section, however, we will show that the sequence \( \{a_m\} \) given in Remark 2.2 satisfies these conditions: let \( n = 0 \) and \( a_0 = a_1 = 1, a_{2m} = a_{2m+1} = 1/(m-1)! \) for all \( m \in \mathbb{N} \) and choose some constants \( \mu > 0 \) and \( \nu = \sqrt{2} \). The second inequality in (3.2) has been verified in Remark 2.2.
The first inequality in (3.2) is also true for all \( m \in \mathbb{N} \) as we see in the following:

\[
\nu^2 |\beta_{2m+3}| = 2 \left| \frac{(-1)^m}{(2m+3)(2m+2)} \frac{1}{m!} \sum_{k=0}^{m-1} (-1)^k k! a_{2k+1} \right| \\
\leq \frac{2}{(2m+3)(2m+2)m} |a_{2m+1}| \left| 1 + \sum_{k=1}^{m-1} (-1)^k k \right| \\
\leq \frac{2}{(2m+3)(2m+2)m} |a_{2m+1}| \left[ \frac{m+1}{2} \right] \\
\leq \frac{m+1}{(2m+3)(2m+2)m} |a_{2m+1}| \\
\leq |a_{2m+1}|,
\]

where \([x]\) denotes the largest integer not exceeding \( x \).

It is not difficult to show that

\[
\frac{1}{4k} \leq \frac{(2k)!}{4^{k+1}(k+1)!} - \frac{(2k)!}{4^k k!(k-1)!} \leq \frac{2k-1}{4k}, \quad (4.2)
\]

\[
\frac{1}{2} \leq \frac{(2k)!}{4^k k!(k-1)!} \leq k - \frac{1}{2} \quad (4.3)
\]

for all \( k \in \mathbb{N} \).

By using (4.2), we will now prove that

\[
\left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} \right| \to \infty \quad (4.4)
\]

as \( m \to \infty \): if \( m = 2\ell \) for some \( \ell \in \mathbb{N} \), then

\[
1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} = \frac{1}{2} + \sum_{i=1}^{\ell-1} \frac{1}{4^{2i+1}} \frac{(4i)!}{(2i)!(2i-1)!} - \frac{1}{2} - \frac{(4i+2)!}{4^{2i+1} (2i+1)!(2i)!} \\
\leq \frac{1}{2} + \sum_{i=1}^{\ell-1} \left( -\frac{1}{8i} \right) \quad (4.5)
\]

\[
\to -\infty \quad \text{as} \quad m \to \infty.
\]
If $m = 2\ell + 1$ for some $\ell \in \mathbb{N}$, then

$$
1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} = 1 + \sum_{i=1}^{\ell} \left[ \frac{1}{4^{2i}} \frac{(4i)!}{(2i)! (2i-1)!} - \frac{1}{4^{2i-1}} \frac{(4i-2)!}{(2i-1)! (2i-2)!} \right]
\geq 1 + \sum_{i=1}^{\ell} \frac{1}{8i-4}
\rightarrow \infty \text{ as } m \rightarrow \infty.
$$

(4.6)

It then follows from (4.3) and (4.4) that

$$
\mu^2 m(2m+2)(2m+1)|a_{2m+2}| = \mu^2 \frac{4^m m!}{(2m-1)!} \left| \sum_{k=0}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} a_{2k} \right|
= \mu^2 \frac{4^m m!(m-1)!}{(2m)!} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} \right|
\geq \mu^2 \frac{1}{m} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} \right|
\geq |a_{2m}|
$$

(4.7)

for all sufficiently large integers $m$, which proves that the sequence $\{a_m\}$ satisfies the second inequality in (3.1).

Finally, we will show that the sequence $\{a_m\}$ satisfies the first inequality in (3.1). It follows from (4.3) that

$$
\nu^2 |a_{2m+2}| = 2 \frac{4^m m!}{(2m+2)!} \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} \right|
= 2 \frac{4^m m!(m-1)!}{(2m+2)!} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} \right|
\leq \frac{4}{(2m+2)(2m+1)} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} \right|
\leq \frac{4}{(2m+2)(2m+1)} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(2k)!}{k!(k-1)!} \right|
$$

for all sufficiently large integers $m$. This completes the proof.
\[ \left| a_{2m} \right| \leq \frac{4}{(2m+2)(2m+1)} \left| a_{2m} \right| \left( 1 + \sum_{k=1}^{m-1} \left( k - \frac{1}{2} \right) \right) \]
\[ = \frac{2m^2 - 4m + 6}{(2m+2)(2m+1)} \left| a_{2m} \right| \]
\[ < \left| a_{2m} \right| \]  
for each \( m \in \mathbb{N} \).

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**References**