Research Article

Notes on $|N,p,q|_k$ Summability Factors of Infinite Series

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New result concerning $|N,p,q|_k$ summability of the infinite series $\sum a_n\lambda_n$ is presented.

1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums $(s_n)$. Let $(T_n)$ denote the sequence of $(N,p,q)$ means of $(s_n)$. The $(N,p,q)$ transform of $(s_n)$ is defined by

$$T_n = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v \sigma_v,$$

where

$$R_n = \sum_{v=0}^{n} p_{n-v} q_v \neq 0, \quad \text{for any } n \quad (p_{-1} = q_{-1} = R_{-1} = 0).$$

Necessary and sufficient conditions for the $(N,p,q)$ method to be regular are

(i) $\lim_{n \to \infty} p_{n-v} q_n / R_n = 0$ for each $v$,

(ii) $\sum_{v=0}^{n} |p_{n-v} q_v| < K|R_n|$, where $K$ is a positive constant independent of $n$. 
The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\varphi_n - \varphi_{n-1}|^k < \infty,$$

where

$$\varphi_n = \frac{1}{p_n} \sum_{v=0}^{n} p_{n-v} S_v,$$  \hspace{1cm} (1.4)

where $P_n = p_1 + p_2 + \cdots + p_n \to \infty$ as $n \to \infty$.

The series $\sum a_n$ is said to be summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty,$$

where

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} S_v, \hspace{1cm} (1.6)$$

and it is said to be summable $|N, p, q|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty,$$

where $T_n$ is as defined by (1.1).

For $k = 1$, $|N, p, q|_k$ summability reduces to $|N, p, q|$ summability.

The series $\sum a_n$ is said to be $(N, p, q)$ bounded or $\sum a_n = O(1)(N, p, q)$ if 

$$t_n = \sum_{v=1}^{n} p_{n-v} q_v s_v = O(R_n) \quad \text{as } n \to \infty. \hspace{1cm} (1.8)$$

By $M$, we denote the set of sequences $p = (p_n)$ satisfying

$$\frac{p_{n+2}}{p_n} \leq \frac{p_{n+1}}{p_{n+2}} \leq 1, \quad p_n > 0, \quad n = 0, 1, \ldots$$

It is known (Das [1]) that for $p \in M$, (1.5) holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n P_n} \left| \sum_{v=1}^{n} p_{n-v} a_v \right| < \infty.$$
For \( p \in M \), the series \( \sum a_n \) is said to be \(|N, p|_k\)-summable, \( k \geq 1 \), (Sulaiman [2]), if
\[
\sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{v=1}^{n} p_{n-v} a_v \right|^k < \infty,
\]
where \( P_n = p_1 + p_2 + \cdots + p_n \to \infty \) as \( n \to \infty \).

It is quite reasonable to give the following definition.
For \( p \in M \), the series \( \sum a_n \) is said to be \(|N, p, q|_k\)-summable, \( k \geq 1 \), if
\[
\sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{v=1}^{n} q_{p_n-v} a_v \right|^k < \infty,
\]
where \( P_n = p_1 + p_2 + \cdots + p_n \to \infty \) as \( n \to \infty \).

We also assume that \((p_n), (q_n)\) are positive sequences of numbers such that
\[
P_n = p_0 + p_1 + \cdots + p_n \to \infty, \quad \text{as } n \to \infty,
\]
\[
Q_n = q_0 + q_1 + \cdots + q_n \to \infty, \quad \text{as } n \to \infty.
\]

A positive sequence \( a = (a_n) \) is said to be a quasi-\( f \)-power increasing sequence, \( f = (f_n) \), if there exists a constant \( K = K(a, f) \) such that
\[
K f_n a_n \geq f_m a_m,
\]
holds for \( n \geq m \geq 1 \) (see [3]).

Das [1], in 1966, proved the following result.

**Theorem 1.1.** Let \((p_n) \in M, q_n \geq 0 \). Then if \( \sum a_n \) is \(|N, p, q|\)-summable, it is \(|N, q_n|\)-summable.

Recently Singh and Sharma [4] proved the following theorem.

**Theorem 1.2.** Let \((p_n) \in M, q_n > 0 \) and let \((q_n)\) be a monotonic nondecreasing sequence for \( n \geq 0 \). The necessary and sufficient condition that \( \sum a_n \lambda_n \) is \(|N, q_n|\)-summable whenever
\[
\sum a_n = O(1)(N, p, q),
\]
\[
\sum_{n=0}^{\infty} \frac{q_n |\lambda_n|}{Q_n} < \infty,
\]
\[
\sum_{n=0}^{\infty} |\Delta \lambda_n| < \infty,
\]
\[
\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} |\Delta^2 \lambda_n| < \infty,
\]
Lemma 2.2. Let $\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n||\lambda_n| < \infty$. \hfill (1.16)

2. Lemmas

Lemma 2.1. Let $(p_n)$ be nonincreasing, $n = O(P_n)$. Then for $r > 0, k \geq 1$,

$$\sum_{n=0}^{\infty} \frac{p_n^{k-r}}{n^r P_n^k} = O\left(\frac{1}{r^{r+k-1}}\right). \hfill (2.1)$$

Proof. Since $p_n$ is nonincreasing, then $np_n = O(P_n)$.

$$\sum_{n=0}^{\infty} \frac{p_n^{k-r}}{n^r P_n^k} = \sum_{n=0}^{\infty} \frac{p_n^{k-r}}{n^r P_n^k} + \sum_{n=0}^{\infty} \frac{p_n^{k-r}}{n^r P_n^k} = O(1) \sum_{n=0}^{\infty} \frac{p_n^{k-r}}{n^r P_n^k} = O(1) \sum_{m=1}^{\infty} \frac{p_m}{m^r P_m^k} = O\left(\frac{1}{r^{r+k-1}}\right). \hfill (2.2)$$

Therefore

$$\sum_{n=0}^{\infty} \frac{p_n^{k-r}}{n^r P_n^k} = O\left(\frac{1}{r^{r+k-1}}\right). \hfill (2.3)$$

□

Lemma 2.2. For $p \in M$,

$$\sum_{n=0}^{\infty} |\Delta_v p_n| < \infty. \hfill (2.4)$$

Proof. Since $p \in M$, then $(p_n)$ is nonincreasing and hence

$$\sum_{n=0}^{m} |\Delta_v p_n| = \sum_{n=0}^{m} (p_{n-v-1} - p_{n-v}) = p_n - p_{m-v-1} = O(1). \hfill (2.5)$$

□
Lemma 2.3 (see [3]). If \((X_n)\) is a quasi-f-increasing sequence, where \(f = (f_n) = (n^\beta (\log n)^\gamma)\), \(\gamma > 0\), \(0 < \beta < 1\), then under the conditions

\[
X_m|\lambda_m| = O(1), \quad m \to \infty,
\]
\[
\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1), \quad m \to \infty,
\]

one has

\[
nX_n|\Delta \lambda_n| = O(1),
\]
\[
\sum_{n=1}^{\infty} X_n|\Delta \lambda_n| < \infty.
\]

3. Result

Our aim is to present the following new general result.

Theorem 3.1. Let \(p \in M\), and let \((X_n)\) be a quasi-f-increasing sequence, where \(f = (f_n) = (n^\beta \log^\gamma n)\), \(\gamma > 0\), \(0 < \beta < 1\) and (2.6), and

\[
\sum_{v=1}^{n} \frac{q_v|s_v|^k}{vX_v^{k-1}} = O(X_n),
\]
\[
\Delta q_v = O\left((v^{-1} q_v)\right),
\]
\[
q_{v+1} = O(q_v),
\]
\[
v = O(P_v),
\]

are all satisfied, then the series \(\sum a_n\lambda_n\) is summable \(|N, p, q|_k\), \(k \geq 1\).

Proof. We have

\[
T_n = \sum_{v=0}^{n} vp_{n-v} q_v a_v \lambda_v
\]
\[
= \sum_{v=0}^{n-1} \left( \sum_{r=0}^{v} a_r \right) \Delta_v (v p_{n-v} q_v \lambda_v) + \left( \sum_{v=0}^{n} a_v \right) n p_0 q_n \lambda_n
\]
\[
= \sum_{v=0}^{n-1} s_v (-p_{n-v} q_v \lambda_v \lambda_v + (v + 1) \Delta q_v p_{n-v} \lambda_v + (v + 1) q_{v+1} \Delta p_{n-v} \lambda_v
\]
\[
+ (v + 1) q_{v+1} p_{n-v-1} \Delta \lambda_v) + n p_0 q_n s_n \lambda_n
\]
\[
= T_{n1} + T_{n2} + T_{n3} + T_{n4} + T_{n5}.
\]
In order to prove the result, it is sufficient, by Minkowski’s inequality, to show that
\[
\sum_{n=1}^{\infty} \frac{1}{nP_n} |T_{nj}|^k < \infty, \quad j = 1, 2, 3, 4, 5.
\] (3.3)

Applying Hölder’s inequality, we have
\[
\sum_{n=1}^{m} \frac{1}{nP_n} |T_{nj}|^k = \sum_{n=1}^{m} \frac{1}{nP_n} \left( \sum_{\nu=0}^{n-1} p_{n-\nu} q_{\nu} s_{\nu} \lambda_{\nu} \right)^k \\
\leq \sum_{n=1}^{m} \frac{1}{nP_n} \sum_{\nu=0}^{n-1} p_{n-\nu} q_{\nu}^k |s_{\nu}|^k |\lambda_{\nu}|^k \left( \sum_{\nu=0}^{n-1} p_{n-\nu} \right)^{k-1} \\
= O(1) \sum_{n=1}^{m} \frac{1}{nP_n} \sum_{\nu=0}^{n-1} p_{n-\nu} q_{\nu}^k |s_{\nu}|^k |\lambda_{\nu}|^k \\
= O(1) \sum_{\nu=0}^{m} q_{\nu}^k |s_{\nu}|^k |\lambda_{\nu}|^k \sum_{\nu=0}^{m} \frac{p_{n-\nu}}{nP_n} \\
= O(1) \sum_{\nu=0}^{m} \frac{q_{\nu}^k |s_{\nu}|^k}{\nu X_{\nu}^{k-1}} |\lambda_{\nu}| |\lambda_{\nu}|^{-1} X_{\nu}^{k-1} \\
= O(1) \sum_{\nu=0}^{m} \frac{q_{\nu}^k |s_{\nu}|^k}{\nu X_{\nu}^{k-1}} |\lambda_{\nu}| \\
= O(1) \sum_{\nu=0}^{m} \frac{q_{\nu}^k |s_{\nu}|^k}{\nu X_{\nu}^{k-1}} |\lambda_{\nu}| \\
= O(1) \sum_{\nu=0}^{m-1} \Delta |\lambda_{\nu}| \sum_{r=0}^{\nu} q_{r}^k |s_{r}|^k r X_{r}^{k-1} + |\lambda_m| \sum_{\nu=0}^{m} \frac{q_{\nu}^k |s_{\nu}|^k}{\nu X_{\nu}^{k-1}} \\
= O(1) \sum_{\nu=0}^{m-1} |\Delta \lambda_{\nu}| X_{\nu} + |\lambda_m| X_{m} = O(1),
\]

\[
\sum_{n=1}^{m} \frac{1}{nP_n} |T_{nj2}|^k = \sum_{n=1}^{m} \frac{1}{nP_n} \left( (\nu + 1)p_{n-\nu} \Delta q_{\nu} s_{\nu} \lambda_{\nu} \right)^k \\
\leq \sum_{n=1}^{m} \frac{1}{nP_n} \sum_{\nu=0}^{n-1} \nu^k p_{n-\nu} |\Delta q_{\nu}|^k |s_{\nu}|^k |\lambda_{\nu}|^k \left( \sum_{\nu=0}^{n-1} p_{n-\nu} \right)^{k-1} \\
= O(1) \sum_{n=1}^{m} \frac{1}{nP_n} \sum_{\nu=0}^{n-1} \nu^k p_{n-\nu} |\Delta q_{\nu}|^k |s_{\nu}|^k |\lambda_{\nu}|^k \\
= O(1) \sum_{\nu=0}^{m} \nu^k |\Delta q_{\nu}|^k |s_{\nu}|^k |\lambda_{\nu}|^k \sum_{\nu=0}^{m} \frac{p_{n-\nu}}{nP_n} 
\]
\[= O(1) \sum_{v=0}^{m} v^{k-1} |\Delta q_v|^{k} |s_v|^{k} |\lambda_v|^{k}\]
\[= O(1) \sum_{v=0}^{m} q_v^{k} |s_v|^{k} \frac{1}{vX_v^{k-1}} |\lambda_v|\]
\[= O(1), \quad \text{as in the case of } T_{n1}, \]
\[\sum_{n=1}^{m} \frac{1}{nP_n^{k}} |T_{n3}|^{k} = \sum_{n=1}^{m} \frac{1}{nP_n^{k}} \sum_{v=0}^{n-1} (v+1) \Delta vP_{n-v} q_{v+1} s_v |\lambda_v|^{k}\]
\[\leq \sum_{n=1}^{m} \frac{1}{nP_n^{k}} \sum_{v=0}^{n-1} v^{k} \Delta vP_{n-v} q_{v+1}^{k} |s_v|^{k} |\lambda_v|^{k} \left( \sum_{v=0}^{n-1} |\Delta vP_{n-v}| \right)^{k-1}\]
\[= O(1) \sum_{v=0}^{m} v^{k} q_{v+1} q_{v+1}^{k} |s_v|^{k} |\lambda_v|^{k}\]
\[= O(1) \sum_{v=0}^{m} v^{k-1} p_v^{k} q_{v+1}^{k} |s_v|^{k} |\lambda_v|^{k}\]
\[= O(1) \sum_{v=0}^{m} v^{k-1} q_{v+1}^{k} |s_v|^{k} |\lambda_v|^{k}\]
\[= O(1), \quad \text{as in the case of } T_{n1}, \]
\[\sum_{n=1}^{m} \frac{1}{nP_n^{k}} |T_{n4}|^{k} = \sum_{n=1}^{m} \frac{1}{nP_n^{k}} \sum_{v=0}^{n-1} (v+1) p_{n-v-1} q_{v+1} s_v \Delta \lambda_v |\lambda_v|^{k}\]
\[\leq \sum_{n=1}^{m} \frac{1}{nP_n^{k}} \sum_{v=0}^{n-1} v^{k} p_{n-v-1} q_{v+1}^{k} |s_v|^{k} |\Delta \lambda_v| X_v^{k-1} \left( \sum_{v=0}^{n-1} X_v |\Delta \lambda_v| \right)^{k-1}\]
\[= O(1) \sum_{v=0}^{m} v^{k} q_{v+1}^{k} |s_v|^{k} |\Delta \lambda_v| X_v^{k-1} \left( \sum_{v=0}^{n-1} X_v |\Delta \lambda_v| \right)^{k-1}\]
\[= O(1) \sum_{v=0}^{m-1} \Delta (v|\Delta \lambda_v|) \sum_{r=0}^{v} \frac{q_r^{k} |s_r|^{k}}{rX_r^{k-1}} + m|\Delta \lambda_m| \sum_{v=0}^{m} \frac{q_v^{k} |s_v|^{k}}{vX_v^{k-1}}\]
\[= O(1) \sum_{v=0}^{m-1} |\Delta \lambda_v| X_v + O(1) \sum_{v=0}^{m-1} |\Delta^2 \lambda_v| X_v + O(1)|\Delta \lambda_m| X_m = O(1),\]
\[
\sum_{n=1}^{m} \frac{1}{n^k} |T_n|^k = \sum_{n=1}^{m} \frac{1}{n^k} |np q_n s_n \lambda_n|^k \\
= O(1) \sum_{n=1}^{m} n^{k-1} p_n^{-k} q_n^k |s_n|^k |\lambda_n|^k \\
= O(1) \sum_{n=1}^{m} n^{-1} q_n^k |s_n|^k |\lambda_n|^k \\
= O(1), \quad \text{as in the case of } T_{n1}.
\] (3.4)

This completes the proof of the theorem.

References