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Iterative algorithms approach to a general system of nonlinear variational inequalities with perturbed mappings and fixed point problems for nonexpansive semigroups

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Abstract

In this paper, we introduce new iterative algorithms for finding a common element of the set of solutions of a general system of nonlinear variational inequalities with perturbed mappings and the set of common fixed points of a one-parameter nonexpansive semigroup in Banach spaces. Furthermore, we prove the strong convergence theorems of the sequence generated by these iterative algorithms under some suitable conditions. The results obtained in this paper extend the recent ones announced by many others.

Mathematics Subject Classification (2010): 47H09, 47J05, 47J25, 49J40, 65J15

Keywords: one-parameter nonexpansive semigroup, perturbed mapping, iterative algorithm, variational inequality, strong convergence, Banach space, common fixed points

1 Introduction

Variational inequality theory has been studied widely in several branches of pure and applied sciences. Indeed, applications of variational inequalities span as diverse disciplines as differential equations, time-optimal control, optimization, mathematical programming, mechanics, finance, and so on (see, e.g., [1,2] for more details). Note that most of the variational problems include minimization or maximization of functions, variational inequality problems, quasivariational inequality problems, decision and management sciences, and engineering sciences problems. For more details, we recommend the reader [3-8,29-31].

Let X be a real Banach space, and X^* be its dual space. The *duality mapping* $J : X \rightarrow 2^{X^*}$ is defined by

$$J(x) = \left\{ f \in X^* : \langle x, f \rangle = ||x||^2, ||f|| = ||x|| \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. If $X : = H$ is a real Hilbert space, then $J = I$ where I is the identity mapping. It is well known that if X is smooth, then J is single-valued, which is denoted by j (see [9]).

Let C be a nonempty closed and convex subset of X and T be a self-mapping of C. We denote $\to$ and $\rightharpoonup$ by strong and weak convergence, respectively. Recall that a mapping $T : C \to C$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$||Tx - T\gamma|| \le L||x - \gamma||, \quad \forall x, \gamma \in C.$$

If $0 < L < 1$, then T is a contraction and if $L = 1$, then T is a nonexpansive mapping. We denote by Fix(T) the set of all fixed points set of the mapping T, i.e., Fix(T) = $\{x \in C : Tx = x\}$.

A mapping $F : C \to X$ is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - F\gamma, j(x - \gamma) \rangle \ge 0, \quad \forall x, \gamma \in C.$$

A mapping $F : C \to X$ is said to be *strongly accretive* if there exists a constant $\eta > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Fx - F\gamma, j(x - \gamma) \rangle \ge \eta ||x - \gamma||^2, \quad \forall x, \gamma \in C.$$

Remark 1.1. If $X := H$ is a real Hilbert space, accretive and strongly accretive mappings coincide with monotone and strongly monotone mappings, respectively.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $||\cdot||$, respectively. Let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \ge \overline{\gamma} ||x||^2, \quad \forall x \in H. \tag{1.1}$$

Remark 1.2. From the definition of operator A, we note that a strongly positive bounded linear operator A is a $||A||$-Lipschitzian and η-strongly monotone operator.

Let C be a nonempty closed and convex subset of a real Banach space X. Recall that the *classical variational inequality* is to find $x^* \in C$ such that

$$\langle \Psi x^*, j(x - x^*) \rangle \ge 0, \quad \forall x \in C, \tag{1.2}$$

where $\Psi: C \to X$ is a nonlinear mapping and $j(x - x^*) \in J(x - x^*)$. The set of solution of variational inequality is denoted by $VI(C, \Psi)$. If $X := H$ is a real Hilbert space, then (1.2) reduces to find $x^* \in C$ such that

$$\langle \Psi x^*, x - x^* \rangle \ge 0, \quad \forall x \in C. \tag{1.3}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonex-pansive mapping on a real Hilbert space H

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle, \tag{1.4}$$

where C is the fixed point set of a nonexpansive mapping T on H and u is a given point in H.

In 2001, Yamada [10] introduced a hybrid steepest descent method for a nonexpansive mapping T as follows:

$$x_{n+1} = Tx_n - \mu \lambda_n F(Tx_n), \quad \forall n \ge 0, \tag{1.5}$$

where F is a κ-Lipschitzian and η-strongly monotone operator with constants κ, $\eta > 0$ and $0 < \mu < \dfrac{2\eta}{\kappa^2}$. He proved that if $\{\lambda_n\}$ satisfying appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(T). \tag{1.6}$$

In 2006, Marino and Xu [11] introduced and considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0, \tag{1.7}$$

where A is a strongly positive bounded linear operator on a real Hilbert space H. They, proved that, if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the variational inequality

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in Fix(T), \tag{1.8}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.9}$$

where C is the fixed point set of a nonexpansive mapping T and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$).

Recently, Tian [12] combined the iterative method (1.7) with the Yamada's method (1.5) and considered the general iterative method for a nonexpansive mapping T as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)Tx_n, \quad \forall n \geq 0. \tag{1.10}$$

Then, he proved that the sequence $\{x_n\}$ generated by (1.10) converges strongly to the unique solution of variational inequality

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in Fix(T). \tag{1.11}$$

Let $\Psi_1, \Psi_2 : C \to X$ be two mappings. Yao et al. [7] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 \Psi_1 y^* + x^* - y^*, j(x - x^*) \rangle \geq 0, \forall x \in C, \\ \langle \rho_2 \Psi_2 x^* + y^* - x^*, j(x - y^*) \rangle \geq 0, \forall x \in C, \end{cases} \tag{1.12}$$

which is called *a general system of nonlinear variational inequalities in Banach spaces*, where $\rho_1 > 0$ and $\rho_2 > 0$ are two constants. In particular, if $\rho_1 = 1$ and $\rho_2 = 1$ then problem (1.12) reduces to problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \Psi_1 y^* + x^* - y^*, j(x - x^*) \rangle \geq 0, \forall x \in C, \\ \langle \Psi_2 x^* + y^* - x^*, j(x - y^*) \rangle \geq 0, \forall x \in C, \end{cases} \tag{1.13}$$

which is defined by Yao et al. [13].

Very recently, Yao et al. [7] introduced an iterative algorithm for solving the problem (1.12). To be more precise, they proved the following theorem.

Theorem YLKY [7]*Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X and let Q_C be a sunny nonexpansive retraction from X onto C. Let the mappings Ψ_1, $\Psi_2 : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $A : C \to X$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Let $\Omega := VI(C, \Psi_1) \cap VI(C, \Psi_2)$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} z_n = Q_C (x_n - \rho_2 \Psi_2 x_n), \\ y_n = Q_C (z_n - \rho_1 \Psi_1 z_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) Q_C (I - \alpha_n A) y_n, \quad \forall n \geq 0. \end{cases} \tag{1.14}$$

Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1] satisfying the following conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < \lim \inf_{n\to\infty} \beta_n \leq \lim \sup_{n\to\infty} \beta_n < 1$.
Then, $\{x_n\}$ converges strongly to $\widehat{x} \in \Omega$ which solves the variational inequality (1.12).

On the other hand, motivated and inspired by the idea of Tian [12] and Yao et al. [7], we consider and introduce the following system of variational inequalities in Banach spaces: Let C be a nonempty closed and convex subset of a Banach space X. Let Ψ_i, $\Phi_i : C \to X$ ($i = 1, 2$) be a mapping. First, we consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 (\Psi_1 + \Phi_1) y^* + x^* - y^*, j (x - x^*) \rangle \geq 0, \forall x \in C, \\ \langle \rho_2 (\Psi_2 + \Phi_2) x^* + y^* - x^*, j (x - y^*) \rangle \geq 0, \forall x \in C, \end{cases} \tag{1.15}$$

which is called *a general system of nonlinear variational inequalities with perturbed mapping in Banach spaces*, where $\rho_1 > 0$ and $\rho_2 > 0$ are two constants. In particular, if $\Phi_1 = \Phi_2 = 0$ then problem (1.15) reduces to problem (1.12). Further, if $\Phi_1 = \Phi_2 = 0$ and $\rho_1 = \rho_2 = 1$ then problem (1.15) reduces to problem (1.13). Second, we introduce iterative algorithms (3.15) below for finding a common element of the set of solutions of a general system of nonlinear variational inequalities with perturbed mappings (1.15) and the set of common fixed points of a one-parameter nonexpansive semigroup in Banach spaces. Furthermore, we show that our iterative algorithm converges strongly to a common element of the two aforementioned sets under some suitable conditions. Our results extend the main result of Tian [12] and Yao et al. [7] and the methods of the proof in this paper are also new and different.

2 Preliminaries

Let $U = \{x \in X : ||x|| = 1\}$. A Banach space X is said to be *strictly convex* if $\frac{||x + y||}{2} < 1$ for all $x, y \in U$ with $x \neq y$. A Banach space X is called *uniformly convex* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$ with $||x||, ||y|| \leq 1$ and $||x - y|| \geq \varepsilon$, $||x + y|| \leq 2(1 - \delta)$ holds. The *modulus of convexity* of X defined by

$$\delta_X (\epsilon) = \inf \left\{ 1 - ||\frac{1}{2} (x + y)|| : ||x||, ||y|| \leq 1, ||x - y|| \geq \epsilon \right\},$$

for all $\varepsilon \in$ [0 2]. It is known that every uniformly convex Banach space is strictly convex and reflexive [9]. The norm of X is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \qquad (2.1)$$

exists for each $x, y \in U$. In this case X is smooth. The norm of X is said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. The norm of X is called *uniformly Fréchet differentiable* if the limit (2.1) is attained uniformly for $x, y \in U$. It is well known that (uniform) *Fréchet differentiability* of the norm of X implies (uniform) *Gâteaux differentiability* of the norm of X.

Let $\rho_X : [0, \infty) \to [0, \infty)$ be the *modulus of smoothness* of X defined by

$$\rho_X(\tau) = \sup \left\{ \frac{1}{2} \left(||x + y|| + ||x - y|| \right) - 1 : x \in U, ||y|| \leq \tau \right\}.$$

A Banach space X is said to be *uniformly smooth* if $\dfrac{\rho_X(t)}{t} \to 0$ as $t \to 0$. Suppose that $q > 1$, then X is said to be *q-uniformly smooth* if there exists $c > 0$ such that $\rho_X(t) \leq ct^q$. It is easy to see that if X is q-uniformly smooth, then $q \leq 2$ and X is uniformly smooth. It is well known that X is uniformly smooth if and only if the norm of X is uniformly Fréchet differentiable and hence the norm of X is Fréchet differentiable, in particular, the norm of X is Fréchet differentiable. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L_p, where $p > 1$. More precisely, L_p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Definition 2.1. A one-parameter family $\mathcal{S} = \{T(t) : t > 0\}$ from C into itself is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t)x = T(s)T(t)x$ for all $x \in C$ and $s, t > 0$;

(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous;

(iv) $||T(t)x - T(t)y|| \leq ||x - y||$ for all $x, y \in C$ and $t > 0$.

Remark 2.2. We denote by $\text{Fix}(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is $\text{Fix}(\mathcal{S}) := \cap_{t>0} \text{Fix}(T(t)) = \{x \in C : T(t)x = x\}$. We know that $\text{Fix}(\mathcal{S})$ is nonempty if C is bounded [14].

Now, we present the concept of a uniformly asymptotically regular semigroup [15-17].

Definition 2.3. Let C be a nonempty closed and convex subset of a Banach space X, $\mathcal{S} = \{T(t) : t > 0\}$ be a continuous operator semigroup on C. Then $\mathcal{S}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on C if for all $h \geq 0$ and any bounded subset B of C such that

$$\lim_{t \to \infty} \sup_{x \in B} ||T(h)T(t)x - T(t)x|| = 0.$$

The nonexpansive semigroup $\{\sigma_t : t > 0\}$ defined by the following lemma is an example of u.a.r. operator semigroup. Other examples of u.a.r. operator semigroup can be found in [15].

Lemma 2.4. [18]*Let C be a nonempty closed and convex subset of a uniformly convex Banach space X, B be a bounded closed and convex subset of C. If we denote $\mathcal{S} = \{T(t) : t > 0\}$ is a nonexpansive semi-group on C such that $\text{Fix}(\mathcal{S}) = \cap_{t>0} \text{Fix}(T(t)) \neq \emptyset$. For all $h \geq 0$, the set $\sigma_t(x) = \dfrac{1}{t} \int_0^t T(s)x \, ds$, then*

$$\lim_{t\to\infty}\sup_{x\in B}\|\sigma_t(x) - T(h)\sigma_t(x)\| = 0.$$

Example 2.5. *The set* $\{\sigma_t : t > 0\}$ *defined by Lemma 2.4 is u.a.r. nonexpansive semigroup. In fact, it is obvious that* $\{\sigma_t : t > 0\}$ *is a nonexpansive semigroup. For each* $h > 0$, *we have*

$$\|\sigma_t(x) - \sigma_h\sigma_t(x)\| = \left\|\sigma_t(x) - \frac{1}{h}\int_0^h T(s)\sigma_t(x)\,\mathrm{d}s\right\|$$

$$= \left\|\frac{1}{h}\int_0^h (\sigma_t(x) - T(s)\sigma_t(x))\,\mathrm{d}s\right\|$$

$$\le \frac{1}{h}\int_0^h \|\sigma_t(x) - T(s)\sigma_t(x)\|\,\mathrm{d}s.$$

By Lemma 2.4, we obtain that

$$\lim_{t\to\infty}\sup_{x\in B}\|\sigma_t(x) - \sigma_h\sigma_t(x)\| \le \frac{1}{h}\int_0^h \lim_{t\to\infty}\sup_{x\in B}\|\sigma_t(x) - T(s)\sigma_t(x)\|\,\mathrm{d}s = 0.$$

Let D be a nonempty subset of C. A mapping $Q : C \to D$ is said to be *sunny* [19] if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \ge 0$. A mapping $Q : C \to D$ is said to be *retraction* if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. It is well known that if $X : = H$ is a real Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C. The following lemmas concern the sunny nonexpansive retraction.

Lemma 2.6. [19]*Let C be a closed and convex subset of a smooth Banach space X. Let D be a nonempty subset of C. Let* $Q : C \to D$ *be a retraction and let J be the normalized duality mapping on X. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive.*

(b) $\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle$, $\forall x, y \in C$.

(c) $\langle x - Qx, J(y - Qx)\rangle \le 0$, $\forall x \in C, y \in D$.

Lemma 2.7. [20]*If X is a strictly convex and uniformly smooth Banach space and if* $T : C \to C$ *is a nonexpansive mapping having a nonempty fixed point set* $\mathrm{Fix}(T)$, *then the set* $\mathrm{Fix}(T)$ *is a sunny nonexpansive retraction of C.*

Let $\mathbb{N}$ be the set of positive integers and let l^∞ be the Banach space of bounded valued functions on $\mathbb{N}$ with supremum norm. Let LIM be a linear continuous functional on l^∞ and let $x = (a_1, a_2,...) \in l^\infty$. Then sometimes, we denote by $\mathrm{LIM}_n(a_n)$ the value of $\mathrm{LIM}(x)$. We know that there exists a linear continuous functional LIM on l^∞ such that $\mathrm{LIM} = \mathrm{LIM}(1) = 1$ and $\mathrm{LIM}(a_n) = \mathrm{LIM}(a_{n+1})$ for each $x = (a_1, a_2,...) \in l^\infty$. Such a LIM is called a *Banach limit*. Let LIM be a Banach limit. Then

$$\liminf_{n\to\infty} a_n \le LIM(x) \le \limsup_{n\to\infty} a_n,$$

Let l^∞ be a Banach space of all bounded real-valued sequences. A Banach limit LIM_n [9] is a linear continuous functional on l^∞ such that

$$||\text{LIM}|| = 1, \quad \liminf_{n\to\infty} a_n \leq \text{LIM}_n a_n \leq \limsup_{n\to\infty} a_n,$$

for each $x = (a_1, a_2,...) \in l^\infty$. Specially, if $a_n \to a$, then $LIM(x) = a$ [9].

In order to prove our main results, we need the following lemmas.

Lemma 2.8. [21]*Let X be a real 2-uniformly smooth Banach space with the best smoothness constant K >0. Then the following inequality holds:*

$$\left\| x + y \right\|^2 \leq \|x\|^2 + 2 \left\langle y, Jx \right\rangle + 2 \left\| Ky \right\|^2, \quad \forall x, y \in X.$$

Lemma 2.9. [22]*In a real Banach space X, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2 \left\langle y, j\left(x + y\right) \right\rangle, \quad \forall x, y \in X,$$

where $j(x + y) \in J(x + y)$.

Lemma 2.10. [23]*Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in [0, 1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n <1$. Suppose $x_{n+1} = (1 - \beta_n)l_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(||l_{n+1} - l_n|| - ||x_{n+1} - x_n||) \leq 0$. Then, $\lim_{n\to\infty} ||l_n - x_n|| = 0$.*

Lemma 2.11. [24]*Let C be a closed and convex subset of a strictly convex Banach space X. Let T_1 and T_2 be two nonexpansive mappings from C into itself with Fix(T_1) $\cap$ Fix(T_2) $= \emptyset$. Define a mapping S by*

$$Sx = \delta T_1 x + (1 - \delta) T_2 x, \quad \forall x \in C,$$

where δ is a constant in (0, 1). Then S is nonexpansive and Fix(S) = Fix(T_1) $\cap$ Fix(T_2).

Lemma 2.12. [9]*Let C be a closed and convex subset of a reflexive Banach space X. Let μ be a proper convex lower semicontinuous function of C into (-∞, ∞] and suppose that $\mu(x_n) \to \infty$ as $||x_n|| \to \infty$. Then, there exists $z \in C$ such that $\mu(z) = \inf_{x\in C}\{\mu(x): x \in C\}$.*

Lemma 2.13. [25]*Let C be a nonempty closed convex subset of a smooth Banach space X and let $\mathcal{S} = \{T(h) : h > 0\}$ be a u.a.r. nonexpansive semigroup on C such Fix$(\mathcal{S}) = \cap_{h>0} Fix(T(h)) \neq \emptyset$ and at least there exists a T(h) which is demicompact. Then, for each $x \in C$, there exists a sequence $\{T(t_k): t_k >0, k \in \mathbb{N}\} \subset \{T(h): h >0\}$ such that $\{T(t_k)x\}$ converges strongly to some point in Fix$(\mathcal{S})$, where $\lim_{k\to\infty} t_k = \infty$.*

Lemma 2.14. [26]*Let C be a nonempty closed and convex subset of a real uniformly convex Banach space and $T : C \to C$ be a nonexpansive mapping such that Fix(T) $\neq \emptyset$. Then, I - T is demiclosed at zero.*

Lemma 2.15. [27]*Let X be a real smooth and uniformly convex Banach space and let r >0. Then there exists a strictly increasing, continuous and convex function g : [0, 2r] $\to \mathbb{R}$ such that g(0) = 0 and*

$$g\left(||x - y||\right) \leq ||x||^2 - 2 \left\langle x, jy \right\rangle + ||y||^2, \quad \forall x, y \in B_r.$$

Lemma 2.16. [28]*Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \sigma_n) a_n + \delta_n,$$

where $\{\sigma_n\}$ is a sequence in (0, 1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \sigma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\sigma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.17. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth Banach space X. Let $F : C \to X$ be a κ-Lipschitzian and η-strongly accretive operator with constants κ, $\eta > 0$. Let $0 < \mu < \dfrac{\eta}{\kappa^2 K^2}$ and $\tau = \mu(\eta - \mu\kappa^2 K^2)$. Then, for each $t \in \left(0, \min\left\{1, \dfrac{1}{2\tau}\right\}\right)$, the mapping $S : C \to C$ defined by $S := (I - t\mu F)$ is contractive with a constant $1 - t\tau$.*

Proof. Since $0 < \mu < \dfrac{\eta}{\kappa^2 K^2}$ and $t \in \left(0, \min\left\{1, \dfrac{1}{2\tau}\right\}\right)$. This implies that $1 - t\tau \in (0, 1)$. From Lemma 2.8, for all $x, y \in C$, we have

$$
\begin{aligned}
||Sx - Sy||^2 &= ||(I - t\mu F)x - (I - t\mu F)y||^2 \\
&= ||x - y - t\mu(Fx - Fy)||^2 \\
&\leq ||x - y||^2 - 2t\mu\langle Fx - Fy, j(x-y)\rangle + 2t^2\mu^2 K^2||Fx - Fy||^2 \\
&\leq ||x - y||^2 - 2t\mu\eta||x - y||^2 + 2t^2\mu^2\kappa^2 K^2||x - y||^2 \\
&\leq \left[1 - 2t\mu\left(\eta - \mu\kappa^2 K^2\right)\right]||x - y||^2 \\
&\leq \left[1 - t\mu\left(\eta - \mu\kappa^2 K^2\right)\right]^2||x - y||^2 \\
&= (1 - t\tau)^2||x - y||^2.
\end{aligned}
$$

It follows that

$$||Sx - Sy|| \leq (1 - t\tau)||x - y||.$$

Hence, we have $S := (I - t\mu F)$ is contractive with a constant $1 - t\tau$. This proof is complete.

Lemma 2.18. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth and uniformly convex Banach space X. Let the mappings Ψ, $\Phi: C \to H$ be $\widetilde{\beta}$-inverse strongly accretive and $\widetilde{\gamma}$-inverse strongly accretive, respectively. Then, we have*

$$||(I - \rho(\Psi + \Phi))x - (I - \rho(\Psi + \Phi))y||^2 \leq ||x-y||^2 + 4\rho^2 K^2\left(\rho - \frac{\widetilde{\beta}}{2K^2}\right)||\Psi x - \Psi y||^2 + 4\rho K^2\left(\rho - \frac{\widetilde{\gamma}}{2K^2}\right)||\Phi x - \Phi y||^2.$$

In particular, if $0 < \rho < \min\left\{\dfrac{\widetilde{\beta}}{2K^2}, \dfrac{\widetilde{\gamma}}{2K^2}\right\}$, then $I - \rho(\Psi + \Phi)$ is nonexpansive.

Proof. By the convexity of $||\cdot||^2$ and Lemma 2.8, for all $x, y \in C$, we have

$$
\begin{aligned}
||(I - \rho(\Psi + \Phi))x - (I - \rho(\Psi + \Phi))y||^2 &= ||x - y - \rho[(\Psi + \Phi)x - (\Psi + \Phi)y]||^2 \\
&= \left\|\frac{1}{2}[x - y - 2\rho(\Psi x + \Phi y)] + \frac{1}{2}[x - y - 2\rho(\Phi x - \Phi y)]\right\|^2 \\
&\leq \frac{1}{2}||x - y - 2\rho(\Psi x - \Psi y)||^2 + \frac{1}{2}||x - y - 2\rho(\Phi x - \Phi y)||^2 \\
&\leq \frac{1}{2}\left[||x - y||^2 - 4\rho\langle\Psi x - \Psi y, j(x-y)\rangle + 8\rho^2 K^2||\Psi x - \Psi y||^2\right] \\
&\quad + \frac{1}{2}\left[||x - y||^2 - 4\rho\langle\Phi x - \Phi y, j(x-y)\rangle + 8\rho^2 K^2||\Phi x - \Phi y||^2\right] \\
&\leq \frac{1}{2}\left[||x - y||^2 - 4\rho\widetilde{\beta}||\Psi x - \Psi y||^2 + 8\rho^2 K^2||\Psi x - \Psi y||^2\right] \\
&\quad + \frac{1}{2}\left[||x - y||^2 - 4\rho\widetilde{\gamma}||\Phi x - \Phi y||^2 + 8\rho^2 K^2||\Phi x - \Phi y||^2\right] \\
&= ||x - y||^2 + 4\rho K^2\left(\rho - \frac{\widetilde{\beta}}{2K^2}\right)||\Psi x - \Psi y||^2 \\
&\quad + 4\rho K^2\left(\rho - \frac{\widetilde{\gamma}}{2K^2}\right)||\Phi x - \Phi y||^2.
\end{aligned}
$$

It is clear that, if $0 < \rho < \min\left\{\dfrac{\widetilde{\beta}}{2K^2}, \dfrac{\widetilde{\gamma}}{2K^2}\right\}$, then $I - \rho(\Psi + \Phi)$ is nonexpansive.
This proof is complete.

Lemma 2.19. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth and uniformly convex Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $\Psi_i : C \to H$ (i = 1, 2) be $\widetilde{\beta}_i$ -inverse-strongly accretive and $\Phi_i : C \to H$ (i = 1, 2) be $\widetilde{\gamma}_i$-inverse-strongly accretive. Let $G : C \to C$ be the mapping defined by*

$$Gx := Q_C\left[Q_C\left(x - \rho_2\left(\Psi + \Phi\right)x\right) - \rho_1\left(\Psi_1 + \Phi_1\right)Q_C\left(x - \rho_2\left(\Psi_2 + \Phi_2\right)x\right)\right], \quad \forall x \in C.$$

If $0 < \rho_1 < \min\left\{\dfrac{\widetilde{\beta}_1}{2K^2}, \dfrac{\widetilde{\gamma}_1}{2K^2}\right\}$ and $0 < \rho_2 < \min\left\{\dfrac{\widetilde{\beta}_2}{2K^2}, \dfrac{\widetilde{\gamma}_2}{2K^2}\right\}$, then $G : C \to C$ is nonexpansive.

Proof. By Lemma 2.18, for all $x, y \in C$, we have

$$
\begin{aligned}
\|Gx - Gy\| &= \|Q_C\left[Q_C\left(x - \rho_2\left(\Psi + \Phi\right)x\right) - \rho_1\left(\Psi_1 + \Phi_1\right)Q_C\left(x - \rho_2\left(\Psi_2 + \Phi_2\right)x\right)\right] \\
&\quad - Q_C\left[Q_C\left(y - \rho_2\left(\Psi + \Phi\right)y\right) - \rho_1\left(\Psi_1 + \Phi_1\right)Q_C\left(y - \rho_2\left(\Psi_2 + \Phi_2\right)y\right)\right]\| \\
&\leq \|\left(I - \rho_1\left(\Psi_1 + \Phi_1\right)\right)Q_C\left(I - \rho_2\left(\Psi_2 + \Phi_2\right)\right)x - \left(I - \rho_1\left(\Psi_1 + \Phi_1\right)\right)Q_C\left(I - \rho_2\left(\Psi_2 + \Phi_2\right)\right)y\| \\
&\leq \|Q_C\left(I - \rho_2\left(\Psi_2 + \Phi_2\right)\right)x - Q_C\left(I - \rho_2\left(\Psi_2 + \Phi_2\right)\right)\| \\
&\leq \|\left(I - \rho_2\left(\Psi_2 + \Phi_2\right)\right)x - \left(I - \rho_2\left(\Psi_2 + \Phi_2\right)\right)\| \\
&\leq \|x - y\|,
\end{aligned}
$$

which implies that $G : C \to C$ is nonexpansive. This proof is complete.

Lemma 2.20. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth and uniformly convex Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $\Psi_i : C \to H$ (i = 1, 2) be $\widetilde{\beta}_i$-inverse-strongly accretive and $\Phi_i : C \to H$ (i = 1, 2) be $\widetilde{\gamma}_i$-inverse-strongly accretive. For given $(x^*, y^*) \in C \times C$ is a solution of the problem (1.15) if and only if $x^* \in Fix(G)$ and $y^* = Q_C\left(x^* - \rho_2(\Psi_2 + \Phi_2)x^*\right)$, where G is the mapping defined as in Lemma 2.19.*

Proof. Let $(x^*, y^*) \in C \times C$ be a solution of the problem (1.15). Then, we can rewrite (1.15) as

$$
\begin{cases}
\left\langle\left(y^* - \rho_1\left(\Psi_1 + \Phi_1\right)y^* - x^*, j\left(x - x^*\right)\right)\right\rangle \leq 0, \forall x \in C, \\
\left\langle\left(x^* - \rho_2\left(\Psi_2 + \Phi_2\right)x^* - y^*, j\left(x - y^*\right)\right)\right\rangle \leq 0, \forall x \in C.
\end{cases}
\tag{2.2}
$$

From Lemma 2.6(c), we can deduce that (2.2) is equivalent to

$$
\begin{cases}
x^* = Q_C\left(y^* - \rho_1\left(\Psi_1 + \Phi_1\right)y^*\right), \\
y^* = Q_C\left(x^* - \rho_2\left(\Psi_2 + \Phi_2\right)x^*\right).
\end{cases}
\tag{2.3}
$$

This proof is complete.

3 Main results

Let C be a nonempty closed and convex subset of a real 2-uniformly smooth and uniformly convex Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $F : C \to X$ be a κ-Lipschitizian and η-strongly accretive operator with constants $\kappa, \eta > 0$, $V : C \to C$ be an L-Lipschitzian mapping with a constant $L \geq 0$ and $T : C \to C$ be a nonexpansive mapping with Fix(T) = Ø. Let $0 < \mu < \dfrac{\eta}{\kappa^2 K^2}$ and $0 \leq \gamma L < \tau$, where

$\tau = \mu(\eta - \mu\kappa^2 K^2)$. For each $t \in \left(0, \min\left\{1, \dfrac{1}{2\tau}\right\}\right)$, consider the mapping $S_t : C \rightarrow C$ defined by

$$S_t x = Q_C\left[t\gamma Vx + (I - t\mu F)\,Tx\right], \quad \forall x \in C.$$

It is easy to see that S_t is contractive. Indeed, from Lemma 2.17, for all $x, y \in C$, we have

$$
\begin{aligned}
||S_t x - S_t y|| &= ||Q_C\left[t\gamma Vx + (I - t\mu F)\,Tx\right] - Q_C\left[t\gamma Vy + (I - t\mu F)\,Ty\right]|| \\
&\leq ||\left[t\gamma Vx + (I - t\mu F)\,Tx\right] - \left[t\gamma Vy + (I - t\mu F)\,Ty\right]|| \\
&= ||t\gamma\left(Vx - Vy\right) + (I - t\mu F)\left(Tx - Ty\right)|| \\
&\leq t\gamma\,||Vx - Vy|| + (1 - t\tau)\,||Tx - Ty|| \\
&\leq (1 - (\tau - \gamma L)\,t)\,||x - y||.
\end{aligned}
$$

Hence S_t is contractive. By the Banach contraction principle, S_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = Q_C\left[t\gamma Vx_t + (I - t\mu F)\,Tx_t\right]. \tag{3.1}$$

Lemma 3.1. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth and uniformly convex Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $F : C \rightarrow X$ be a κ-Lipschitizian and η-strongly accretive operator with constants κ, η >0, $V : C \rightarrow C$ be an L-Lipschizian mapping with a constant $L \geq 0$ and $T : C \rightarrow C$ be a nonexpansive mapping with $\mathrm{Fix}(T) = \emptyset$. Let $0 < \mu < \dfrac{\eta}{\kappa^2 K^2}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \mu\kappa^2 K^2)$. Then the net $\{x_t\}$ defined by (3.1) converges strongly to $\widehat{x} \in \mathrm{Fix}\,(T)$ as $t \rightarrow 0$, where $\widehat{x}$ is the unique solution of the variational inequality*

$$\left\langle(\mu F - \gamma V)\widehat{x}, j\,(\widehat{x} - v)\right\rangle \leq 0, \quad \forall v \in Fix\,(T). \tag{3.2}$$

Proof. We observe that

$$
\begin{aligned}
\mu\kappa^2 K^2 > 0 &\Leftrightarrow \eta - \mu\kappa^2 K^2 < \eta \\
&\Leftrightarrow \mu\left(\eta - \mu\kappa^2 K^2\right) < \mu\eta \\
&\Leftrightarrow \tau < \mu\eta.
\end{aligned}
$$

It follows that

$$0 \leq \gamma L < \tau < \mu\eta. \tag{3.3}$$

First, we show the uniqueness of a solution of the variational inequality (3.2). Suppose that $\widehat{x}$, $\tilde{x} \in \mathrm{Fix}\,(T)$ are solution of (3.2), then

$$\left\langle(\mu F - \gamma V)\widehat{x}, j\,(\widehat{x} - \tilde{x})\right\rangle \leq 0 \tag{3.4}$$

and

$$\left\langle(\mu F - \gamma V)\tilde{x}, j\,(\tilde{x} - \widehat{x})\right\rangle \leq 0. \tag{3.5}$$

Adding up (3.4) and (3.5), we have

$$
\begin{aligned}
0 &\geq \langle (\mu F - \gamma V)\widehat{x} - (\mu F - \gamma V)\widetilde{x}, j(\widehat{x} - \widetilde{x}) \rangle \\
&= \mu \langle F\widehat{x} - F\widetilde{x}, j(\widehat{x} - \widetilde{x}) \rangle - \gamma \langle V\widehat{x} - V\widetilde{x}, j(\widehat{x} - \widetilde{x}) \rangle \\
&\geq \mu\eta \|\widehat{x} - \widetilde{x}\|^2 - \gamma \|V\widehat{x} - V\widetilde{x}\|\|\widehat{x} - \widetilde{x}\| \\
&\geq (\mu\eta - \gamma L) \|\widehat{x} - \widetilde{x}\|^2.
\end{aligned}
$$

Note that (3.3) implies that $\widehat{x} = \widetilde{x}$ and the uniqueness is proved. Below, we use $\widehat{x}$ to denote the unique solution of (3.2).

Next, we show that $\{x_t\}$ is bounded. Without loss of generality, we may assume that $t \leq \min\left\{1, \dfrac{1}{2\tau}\right\}$. Take $p \in \text{Fix}(T)$. From Lemma 2.17, we have

$$
\begin{aligned}
\|x_t - p\| &= \|Q_C[t\gamma Vx_t + (I - t\mu F)Tx_t] - Q_C p\| \\
&\leq \|t(\gamma Vx_t - \mu Fp) + (I - t\mu F)(Tx_t - p)\| \\
&\leq t\gamma \|Vx_t - Vp\| + t\|\gamma Vp - \mu Fp\| + (1 - t\tau)\|Tx_t - p\| \\
&\leq (1 - (\tau - \gamma L)t)\|x_t - p\| + t\|\gamma Vp - \mu Fp\|.
\end{aligned}
$$

It follows that

$$
\|x_t - p\| \leq \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma L}.
$$

Hence, $\{x_t\}$ is bounded, so are $\{Vx_t\}$ and $\{FTx_t\}$. Assume $\{t_n\} \subset (0, 1)$ is such that $t_n \to 0$ as $n \to \infty$. Set $x_n := x_{t_n}$. Define a mapping $\varphi : C \to \mathbb{R}$ by

$$
\phi(x) := LIM_n \|x_n - x\|^2, \quad \forall x \in C,
$$

where LIM_n is a Banach limit on l^∞. Note that X is reflexive and φ is continuous, convex functional and $\varphi(x) \to \infty$ as $\|x\| \to \infty$. From Lemma 2.12, there exists $z \in C$ such that $\varphi(z) = \inf_{x \in C} \varphi(x)$. This implies that the set

$$
K := \left\{z \in C : \phi(z) = \inf_{x \in C} \phi(x)\right\} \neq \emptyset.
$$

Observe that

$$
\begin{aligned}
\|x_n - Tx_n\| &= \|Q_C[t_n \gamma Vx_n + (I - t_n \mu F)Tx_n] - Q_C Tx_n\| \\
&\leq t_n \|\gamma Vx_n - \mu FTx_n\| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.6}
$$

For all $z \in C$, we have

$$
\begin{aligned}
\phi(Tz) &= LIM_n \|x_n - Tz\|^2 \\
&= LIM_n \|Tx_n - Tz\|^2 \\
&\leq LIM_n \|x_n - z\|^2 \\
&= \phi(z),
\end{aligned}
$$

which implies that $T(K) \subseteq K$; that is, K is invariant under T. Since X is a uniformly smooth Banach space, it has the fixed point property for nonexpansive mapping T. Then, there exists $\widetilde{x} \in K$ such that $T\widetilde{x} = \widetilde{x}$. Since $\widetilde{x}$ is also minimization of φ over C, it follows that $x \in C$ and $t \in (0, 1)$,

$$
\phi(\widetilde{x}) \leq \phi(\widetilde{x} + t(x - \mu F\widetilde{x})).
$$

From Lemma 2.9, we have

$$||x_n - \widetilde{x} + t\,(\mu F\widetilde{x} - x)\,||^2 \leq ||x_n - \widetilde{x}||^2 + 2t\,\langle \mu F\widetilde{x} - x, j\,(x_n - \widetilde{x} + t\,(\mu F\widetilde{x} - x))\rangle.$$

Taking Banach limit over $n \geq 1$, then

$$\mathrm{LIM}_n||x_n - \widetilde{x} + t\,(\mu F\widetilde{x} - x)\,||^2 \leq \mathrm{LIM}_n||x_n - \widetilde{x}||^2 + 2t\mathrm{LIM}_n\,\langle \mu F\widetilde{x} - x, j\,(x_n - \widetilde{x} + t\,(\mu F\widetilde{x} - x))\rangle,$$

which in turn implies that

$$2t\mathrm{LIM}_n\,\langle x - \mu F\widetilde{x}, j\,(x_n - \widetilde{x} + t\,(\mu F\widetilde{x} - x))\rangle \leq \mathrm{LIM}_n||x_n - \widetilde{x}||^2 - \mathrm{LIM}_n||x_n - \widetilde{x} + t\,(\mu F\widetilde{x} - x)\,||^2$$
$$\leq 0,$$

and hence

$$\mathrm{LIM}_n\,\langle x - \mu F\widetilde{x}, j\,(x_n - \widetilde{x} + t\,(\mu F\widetilde{x} - x))\rangle \leq 0.$$

Again since X is a uniformly smooth Banach space, we have that the duality mapping j is norm-to-norm uniformly continuous on a bounded subset of C (see [9], Lemma 1), letting $t \to 0$, we obtain

$$\mathrm{LIM}_n\,\langle x - \mu F\widetilde{x}, j\,(x_n - \widetilde{x})\rangle \leq 0, \quad \forall x \in C.$$

In particular

$$\mathrm{LIM}_n\,\langle \gamma V\widetilde{x} - \mu F\widetilde{x}, j\,(x_n - \widetilde{x})\rangle \leq 0. \tag{3.7}$$

Set $x_t = Q_C y_t$, where $y_t = t\gamma V x_t + (I - t\mu F)T x_t$. Notice that $x_n := x_{t_n}$ and $y_n := y_{t_n}$. For $\widetilde{x} \in \mathrm{Fix}\,(T)$, by Lemma 2.6(c), we have

$$
\begin{aligned}
||x_n - \widetilde{x}||^2 &= \langle y_n - \widetilde{x}, j\,(x_n - \widetilde{x})\rangle + \langle Q_C y_n - y_n, j\,(Q_C y_n - \widetilde{x})\rangle \\
&\leq \langle y_n - \widetilde{x}, j\,(x_n - \widetilde{x})\rangle \\
&= t_n\,\langle \gamma V x_n - \mu F\widetilde{x}, j\,(x_n - \widetilde{x})\rangle + \langle (I - t_n\mu F)\,(T x_n - \widetilde{x}), j\,(x_n - \widetilde{x})\rangle \\
&\leq t_n\,\langle \gamma V x_n - \mu F\widetilde{x}, j\,(x_n - \widetilde{x})\rangle + (1 - t_n\tau)\,||x_n - \widetilde{x}||^2.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
||x_n - \widetilde{x}||^2 &\leq \frac{1}{\tau}\,\langle \gamma V x_n - \mu F\widetilde{x}, j\,(x_n - \widetilde{x})\rangle \\
&= \frac{1}{\tau}\,\{\gamma\,\langle V x_n - V\widetilde{x}, j\,(x_n - \widetilde{x})\rangle + \langle \gamma V\widetilde{x} - \mu F\widetilde{x}, j\,(x_n - \widetilde{x})\rangle\} \\
&\leq \frac{1}{\tau}\,\{\gamma L||x_n - \widetilde{x}||^2 + \langle \gamma V\widetilde{x} - \mu F\widetilde{x}, j\,(x_n - \widetilde{x})\rangle\},
\end{aligned}
$$

which implies that

$$||x_n - \widetilde{x}||^2 \leq \frac{1}{\tau - \gamma L}\,\langle \gamma V\widetilde{x} - \mu F\widetilde{x}, j\,(x_n - \widetilde{x})\rangle.$$

It follows from (3.7) that

$$\mathrm{LIM}_n||x_n - \widetilde{x}||^2 \leq \frac{1}{\tau - \gamma L}\mathrm{LIM}_n\,\langle \gamma V\widetilde{x} - \mu F\widetilde{x}, j\,(x_n - \widetilde{x})\rangle \leq 0. \tag{3.8}$$

This implies that $\mathrm{LIM}_n||x_n - \widetilde{x}|| = 0$. Hence, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \widetilde{x}$ as $i \to \infty$. From (3.6) and Lemma 2.14, we get $\widetilde{x} \in \mathrm{Fix}\,(T)$.

Next, we show that $\widetilde{x}$ solves the variational inequality (3.2). We note that

$$x_t = Q_C \gamma_t = Q_C \gamma_t - \gamma_t + t\gamma V x_t + (I - t\mu F)\, T x_t,$$

which derives that

$$(\mu F - \gamma V)\, x_t = \frac{1}{t}\left(Q_C \gamma_t - \gamma_t\right) - \frac{1}{t}\,(I - T)\, x_t + \mu\,(F x_t - F T x_t). \tag{3.9}$$

Note that $I - T$ is accretive (i.e., $\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq 0$, for $x, y \in C$). For all $v \in \mathrm{Fix}(T)$, it follows from (3.9) and Lemma 2.6(c) that

$$
\begin{aligned}
\langle (\mu F - \gamma V)\, x_t, j\,(x_t - v)\rangle &= \frac{1}{t}\langle Q_C \gamma_t - \gamma_t, j\left(Q_C \gamma_t - v\right)\rangle - \frac{1}{t}\langle (I - T)\, x_t - (I - T)\, v, j\,(x_t - v)\rangle \\
&\quad + \mu\,\langle F x_t - F T x_t, j\,(x_t - v)\rangle \\
&\leq \mu\,\langle F x_t - F T x_t, j\,(x_t - v)\rangle \\
&\leq \mu ||F x_t - F T x_t||\,||x_t - v|| \\
&\leq ||x_t - T x_t||M,
\end{aligned}
\tag{3.10}
$$

where $M = \sup_{n \geq 1}\{\mu\kappa ||x_t - v||\}$ and $t \in \left(0, \min\left\{1, \frac{1}{2\tau}\right\}\right)$. Now, replacing t in (3.10) with t_n and taking the limit as $n \to \infty$, we noticing that $x_{t_n} - T x_{t_n} \to \widetilde{x} \to T\widetilde{x} = 0$ for $\widetilde{x} \in \mathrm{Fix}\,(T)$, we obtain $\langle (\mu F - \gamma V)\,\widetilde{x}, j\,(\widetilde{x} - v)\rangle \leq 0$. Hence $\widetilde{x} \in \mathrm{Fix}\,(T)$ is the solution of the variational inequality (3.2). Consequently, $\widehat{x} = \widetilde{x}$ by uniqueness. Therefore, $x_t \to \widehat{x}$ as $t \to 0$. This completes the proof.

Lemma 3.2. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth and uniformly convex Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $F : C \to X$ be a κ-Lipschitizian and η-strongly accretive operator with constants $\kappa, \eta > 0$, $V : C \to C$ be an L-Lipschitzian mapping with a constant $L \geq 0$ and $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) = \emptyset$. Let $0 < \mu < \frac{\eta}{\kappa^2 K^2}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \mu\kappa^2 K^2)$. Assume that the net $\{x_t\}$ defined by (3.1) converges strongly to $\widehat{x} \in \mathrm{Fix}\,(T)$ as $t \to 0$. Suppose that $\{x_n\}$ is bounded and $\lim_{n \to \infty} ||x_n - T x_n|| = 0$. Then*

$$\limsup_{n \to \infty} \langle \gamma V\widehat{x} - \mu F\widehat{x}, j\,(x_n - \widehat{x})\rangle \leq 0. \tag{3.11}$$

Proof. Notice that $x_t = Q_C\, y_t$, where $y_t = t\gamma V x_t + (I - t\mu F) T x_t$. We note that

$$
\begin{aligned}
y_t - x_n &= t\gamma V x_t + (I - t\mu F)\, T x_t - x_n \\
&= t\,(\gamma V x_t - \mu F x_t) + T x_t - x_n + t\mu\,(F x_t - F T x_t) \\
&= t\,(\gamma V x_t - \mu F x_t) + (T x_t - T x_n) + (T x_n - x_n) + t\mu\left(F Q_C \gamma_t - F Q_C T x_t\right).
\end{aligned}
$$

It follows from Lemma 2.6(c) that

$$
\begin{aligned}
||x_t - x_n||^2 &= \langle y_t - x_n, j\,(x_t - x_n)\rangle + \langle Q_C \gamma_t - \gamma_t, j\left(Q_C \gamma_t - x_n\right)\rangle \\
&\leq \langle y_t - x_n, j\,(x_t - x_n)\rangle \\
&= t\,\langle \gamma V x_t - \mu F x_t, j\,(x_t - x_n)\rangle + \langle T x_t - T x_n, j\,(x_t - x_n)\rangle + \langle T x_n - x_n, j\,(x_t - x_n)\rangle \\
&\quad + t\mu\,\langle F Q_C \gamma_t - F Q_C T x_t, j\,(x_t - x_n)\rangle \\
&\leq t\,\langle \gamma V x_t - \mu F x_t, j\,(x_t - x_n)\rangle + ||x_t - x_n||^2 + ||T x_n - x_n||\,||x_t - x_n|| \\
&\quad + t\mu\kappa ||Q_C \gamma_t - Q_C T x_t||\,||x_t - x_n|| \\
&\leq t\,\langle \gamma V x_t - \mu F x_t, j\,(x_t - x_n)\rangle + ||x_t - x_n||^2 + ||T x_n - x_n||\,||x_t - x_n|| \\
&\quad + t^2\mu\kappa ||\gamma V x_t - \mu F T x_t||\,||x_t - x_n||,
\end{aligned}
$$

which in turn implies that

$$\langle \gamma V x_t - \mu F x_t, j(x_n - x_t) \rangle \le \frac{||T x_n - x_n||}{t} ||x_t - x_n|| + t \mu \kappa ||\gamma V x_t - \mu F T x_t||. \quad (3.12)$$

Since $x_n - Tx_n \to 0$ as $n \to \infty$, taking the upper limit as $n \to \infty$ firstly, and then as $t \to 0$ in (3.12), we have

$$\limsup_{t \to 0} \limsup_{n \to \infty} \langle \gamma V x_t - \mu F x_t, j(x_n - x_t) \rangle \le 0. \quad (3.13)$$

On the other hand, we note that

$$
\begin{aligned}
\langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - \widehat{x}) \rangle &= \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - \widehat{x}) \rangle - \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - x_t) \rangle \\
&\quad + \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - x_t) \rangle - \langle \gamma V \widehat{x} - \mu F x_t, j(x_n - x_t) \rangle \\
&\quad + \langle \gamma V \widehat{x} - \mu F x_t, j(x_n - x_t) \rangle - \langle \gamma V x_t - \mu F x_t, j(x_n - x_t) \rangle \\
&\quad + \langle \gamma V x_t - \mu F x_t, j(x_n - x_t) \rangle \\
&= \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - \widehat{x}) - j(x_n - x_t) \rangle + \mu \langle F x_t - F \widehat{x}, j(x_n - x_t) \rangle \\
&\quad + \gamma \langle V \widehat{x} - V x_t, j(x_n - x_t) \rangle + \langle \gamma V x_t - \mu F x_t, j(x_n - x_t) \rangle.
\end{aligned}
$$

Taking the upper limit as $n \to \infty$, we have

$$
\begin{aligned}
\limsup_{n \to \infty} \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - \widehat{x}) \rangle &\le \limsup_{n \to \infty} \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - \widehat{x}) - j(x_n - x_t) \rangle \\
&\quad + (\mu \kappa + \gamma L) ||x_t - \widehat{x}|| \limsup_{n \to \infty} ||x_n - x_t|| \\
&\quad + \limsup_{n \to \infty} \langle \gamma V x_t - \mu F x_t, j(x_n - x_t) \rangle.
\end{aligned}
$$

Since X is a uniformly smooth Banach space, we have that the duality mapping j is norm-to-norm uniformly continuous on bounded subset of C (see [9], Lemma 1), then

$$\limsup_{t \to 0} \limsup_{n \to \infty} \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - \widehat{x}) - j(x_n - x_t) \rangle = 0. \quad (3.14)$$

Then, from (3.13) and (3.14), we have

$$
\begin{aligned}
\limsup_{n \to \infty} \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - \widehat{x}) \rangle &= \limsup_{t \to 0} \limsup_{n \to \infty} \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(x_n - \widehat{x}) \rangle \\
&\le \limsup_{t \to 0} \limsup_{n \to \infty} \langle \gamma V x_t - \mu F x_t, j(x_n - x_t) \rangle \\
&\le 0.
\end{aligned}
$$

This completes the proof.

Theorem 3.3. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth and uniformly convex Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $F : C \to X$ be a κ-Lipschitizian and η-strongly accretive operator with constants κ, $\eta > 0$, $V : C \to C$ be an L-Lipschitzian mapping with a constant $L \ge 0$. Let $0 < \mu < \frac{\eta}{\kappa^2 K^2}$ and $0 \le \gamma L < \tau$, where $\tau = \mu(\eta - \mu \kappa^2 K^2)$. Let $\mathcal{S} = \{T(h) : h > 0\}$ be a u.a.r. nonexpansive semigroup from C into itself such that $\mathrm{Fix}(\mathcal{S}) := \cap_{h>0} \mathrm{Fix}(T(h)) \ne \emptyset$ and least there exists a $T(h)$ which is demicompact. Let $\Psi_i : C \to X$ ($i = 1, 2$) be $\widetilde{\beta}_i$-inverse-strongly accretive and $\Phi_i : C \to X$ ($i = 1, 2$) be $\widetilde{\gamma}_i$-inverse-strongly accretive. Assume that $\Omega := \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(G) \ne \emptyset$, where G is defined as in Lemma 2.19. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$
\begin{cases}
z_n = Q_C(x_n - \rho_2(\Psi_2 + \Phi_2) x_n), \\
y_n = Q_C(z_n - \rho_1(\Psi_1 + \Phi_1) z_n), & (3.15) \\
x_{n+1} = \beta_n x_n + (1 - \beta_n) Q_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T(t_n) y_n], \quad \forall n \ge 1,
\end{cases}
$$

where $\rho_i \in \left(0, \min\left\{\dfrac{\widetilde{\beta}_i}{2K^2}, \dfrac{\widetilde{\gamma}_i}{2K^2}\right\}\right)$ for all $i = 1, 2$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{t_n\}$ is a sequence in $(0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(C3) $t_{n+1} = h + t_n$ for all $h > 0$ and $\lim_{n\to\infty} t_n = \infty$.

Then, the sequence $\{x_n\}$ defined by (3.15) converges strongly to $\widehat{x} \in \Omega$ as $n \to \infty$, which $\widetilde{x}$ is the unique solution of the variational inequality

$$\langle (\mu F - \gamma V)\widetilde{x}, j(\widetilde{x} - v) \rangle \le 0, \quad \forall v \in \Omega, \tag{3.16}$$

and $(\widehat{x}, \widehat{y})$ is the solution of the problem (1.15), where $\widehat{y} = Q_C(\widehat{x} - \rho_2(\Psi_2 + \Phi_2)\widehat{x})$.

Proof. Note that from the condition (C1), we assume without loss generality, that $\alpha_n \le \min\left\{1, \dfrac{1}{2\tau}\right\}$ for all $n \ge 1$. First, we show that $\{x_n\}$ is bounded. Take $x^* \in \Omega$. It follows from Lemma 2.19 that

$$x^* = Q_C[Q_C(x - \rho_2(\Psi_2 + \Phi_2)x^*) - \rho_1(\Psi_1 + \Phi_1)Q_C(x^* - \rho_2(\Psi_2 + \Phi_2)x^*)].$$

Put $y^* = Q_C(x^* - \rho_2(\Psi_2 + \Phi_2)x^*)$, then $x^* = Q_C(y^* - \rho_1(\Psi_1 + \Phi_1)y^*)$. From (3.15), we observe that

$$\begin{aligned}
||y_n - x^*|| &= ||Gx_n - Gx^*|| \\
&\le ||x_n - x^*||.
\end{aligned}$$

Set $u_n = Q_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)T(t_n)y_n]$. From Lemma 2.17, we have

$$\begin{aligned}
||u_n - x^*|| &= ||Q_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)T(t_n)y_n\right] - Q_C x^*|| \\
&\le ||\alpha_n(\gamma V x_n - \mu F x^*) + (I - \alpha_n \mu F)(T(t_n)y_n - x^*)|| \\
&\le \alpha_n \gamma ||V x_n - V x^*|| + \alpha_n ||\gamma V x^* - \mu F x^*|| + (1 - \alpha_n \tau)||y_n - x^*|| \\
&\le (1 - (\tau - \gamma L)\alpha_n)||x_n - x^*|| + \alpha_n ||\gamma V x^* - \mu F x^*||.
\end{aligned}$$

It follows that

$$\begin{aligned}
||x_{n+1} - x^*|| &= ||\beta_n(x_n - x^*) + (1 - \beta_n)(u_n - x^*)|| \\
&\le \beta_n ||x_n - x^*|| + (1 - \beta_n)||u_n - x^*|| \\
&\le \beta_n ||x_n - x^*|| + (1 - \beta_n)\left[(1 - (\tau - \gamma L)\alpha_n)||x_n - x^*|| + \alpha_n ||\gamma V x^* - \mu F x^*||\right] \\
&= (1 - (\tau - \gamma L)\alpha_n(1 - \beta_n))||x_n - x^*|| + \alpha_n(1 - \beta_n)(\tau - \gamma L)\frac{||\gamma V x^* - \mu F x^*||}{\tau - \gamma L}.
\end{aligned}$$

By induction, we have

$$||x_n - x^*|| \le \max\left\{||x_1 - x^*||, \frac{||\gamma V x^* - \mu F x^*||}{\tau - \gamma L}\right\}, \quad \forall n \ge 1.$$

Hence, $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{y_n\}$ and $\{u_n\}$.

Next, we show that $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$. We observe that

$$\begin{aligned}
||y_{n+1} - y_n|| &= ||Gx_{n+1} - Gx_n|| \\
&\le ||x_{n+1} - x^*||.
\end{aligned} \tag{3.17}$$

Now, we can take a constant $M > 0$ such that

$$M = \sup_{n \ge 1}\left\{\gamma ||V x_{n+1}|| + \mu ||FT(t_{n+1})y_{n+1}||, \gamma ||V x_n|| + \mu ||FT(t_n)y_n||\right\}.$$

Then, we have

$$
\begin{aligned}
||u_{n+1} - u_n|| &= ||Q_C\left[\alpha_{n+1}\gamma V x_{n+1} + (I - \alpha_{n+1}\mu F) T(t_{n+1}) \gamma_{n+1}\right] - Q_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T(t_n) \gamma_n\right]|| \\
&\leq ||\alpha_{n+1}\gamma V x_{n+1} + (I - \alpha_{n+1}\mu F) T(t_{n+1}) \gamma_{n+1} - \alpha_n \gamma V x_n - (I - \alpha_n \mu F) T(t_n) \gamma_n|| \\
&\leq \alpha_{n+1}||\gamma V x_{n+1} - \mu F T(t_{n+1}) \gamma_{n+1}|| + \alpha_n||\gamma V x_n - \mu F T(t_n) \gamma_n|| + ||T(t_{n+1}) \gamma_{n+1} - T(t_n) \gamma_n|| \\
&\leq (\alpha_{n+1} + \alpha_n) M + ||\gamma_{n+1} - \gamma_n|| + ||T(h) T(t_n) \gamma_n - T(t_n) \gamma_n||.
\end{aligned}
\tag{3.18}
$$

Combining (3.17) and (3.18), we obtain

$$
||u_{n+1} - u_n|| - ||x_{n+1} - x_n|| \leq (\alpha_{n+1} + \alpha_n) M + ||T(h) T(t_n) \gamma_n - T(t_n) \gamma_n||. \tag{3.19}
$$

Since $\{T(h): h > 0\}$ is a u.a.r. nonexpansive semigroup and $\lim_{n\to\infty} t_n = \infty$, then for all $h > 0$, and for any bounded subset B of C containing $\{x_n\}$ and $\{y_n\}$, we obtain that

$$
\lim_{n\to\infty} ||T(h) T(t_n) \gamma_n - T(t_n) \gamma_n|| \leq \lim_{n\to\infty} \sup_{\omega\in B} ||T(h) T(t_n)\omega - T(t_n)\omega = 0. \tag{3.20}
$$

Consequently, it follows from the conditions (*C*1), (*C*2) and (3.19) that

$$
\limsup_{n\to\infty} (||u_{n+1} - u_n|| - ||x_{n+1} - x_n||) \leq 0.
$$

Hence, by Lemma 2.10, we obtain that

$$
\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{3.21}
$$

Consequently, we have

$$
\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} (1 - \beta_n) ||u_n - x_n|| = 0. \tag{3.22}
$$

We observe that

$$
\begin{aligned}
||u_n - T(t_n) \gamma_n|| &= ||Q_C\left[\alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T(t_n) \gamma_n\right] - Q_C T(t_n) \gamma_n|| \\
&\leq \alpha_n ||\gamma f(x_n) - \mu F T(t_n) \gamma_n|| \to 0 \text{ as } n \to \infty.
\end{aligned}
\tag{3.23}
$$

From (3.21) and (3.23), we have

$$
||x_n - T(t_n) \gamma_n|| \leq ||x_n - u_n|| + ||u_n - T(t_n) \gamma_n|| \to 0 \text{ as } n \to \infty. \tag{3.24}
$$

Next, we show that $\lim_{n\to\infty} ||x_n - T(h)x_n|| = 0$, $\forall h > 0$. For all $x^*, y^* \in \Omega$, by the convexity of $||\cdot||^2$ and Lemma 2.8, we have

$$
\begin{aligned}
||z_n - \gamma^*||^2 &= ||Q_C(x_n - \rho_2 (\Psi_2 + \Phi_2) x_n) - Q_C(x^* - \rho_2(\Psi_2 + \Phi_2) x^*)||^2 \\
&\leq ||(x_n - \rho_2 (\Psi_2 + \Phi_2) x_n) - (x^* - \rho_2(\Psi_2 + \Phi_2) x^*)||^2 \\
&= ||x_n - x^* - \rho_2 [(\Psi_2 + \Phi_2) x_n - (\Psi_2 + \Phi_2) x^*]||^2 \\
&= ||\tfrac{1}{2} [x_n - x^* - 2\rho_2 (\Psi_2 x_n - \Psi_2 x^*)] + \tfrac{1}{2} [x_n - x^* - 2\rho_2 (\Phi_2 x_n - \Phi_2 x^*)]||^2 \\
&\leq \tfrac{1}{2}||x_n - x^* - 2\rho_2 (\Psi_2 x_n - \Psi_2 x^*)||^2 + \tfrac{1}{2}||x_n - x^* - 2\rho_2 (\Phi_2 x_n - \Phi_2 x^*)||^2 \\
&\leq \tfrac{1}{2}\left(||x_n - x^*||^2 - 4\rho_2 \langle \Psi_2 x_n - \Psi_2 x^*, x_n - x^* \rangle + 8\rho_2^2 K^2 ||\Psi_2 x_n - \Psi_2 x^*||^2\right) \\
&\quad + \tfrac{1}{2}\left(||x_n - x^*||^2 - 4\rho_2 \langle \Phi_2 x_n - \Phi_2 x^*, x_n - x^* \rangle + 8\rho_2^2 K^2 ||\Phi_2 x_n - \Phi_2 x^*||^2\right) \\
&\leq \tfrac{1}{2}\left(||x_n - x^*||^2 - 4\rho_2 \widetilde{\beta}_2 ||\Psi_2 x_n - \Psi_2 x^*||^2 + 8\rho_2^2 K^2 ||\Psi_2 x_n - \Psi_2 x^* x^*||^2\right) \\
&\quad + \tfrac{1}{2}\left(||x_n - x^*|| - 4\rho_2 \widetilde{\gamma}_2 ||\Phi_2 x_n - \Phi_2 x^*||^2 + 8\rho_2^2 K^2 ||\Phi_2 x_n - \Phi_2 x_n x^*||^2\right) \\
&= ||x_n - x^*||^2 + 4\rho_2 K^2 \left(\rho_2 - \frac{\widetilde{\beta}_2}{2K^2}\right) ||\Psi_2 x_n - \Psi_2 x^*||^2 + 4\rho_2 K^2 \left(\rho_2 - \frac{\widetilde{\gamma}_2}{2K^2}\right) ||\Phi_2 x_n - \Phi_2 x^*||^2.
\end{aligned}
\tag{3.25}
$$

In a similar way, we can get

$$
||\gamma_n - x^*||^2 \leq ||z_n - \gamma^*||^2 + 4\rho_1 K^2 \left(\rho_1 - \frac{\widetilde{\beta}_1}{2K^2}\right) ||\Psi_1 z_n - \Psi_1 \gamma^*||^2 + 4\rho_1 K^2 \left(\rho_1 - \frac{\widetilde{\gamma}_1}{2K^2}\right) ||\Phi_1 z_n - \Phi_1 \gamma^*||^2. \tag{3.26}
$$

Substituting (3.25) into (3.26), we have

$$||\gamma_n - x^*||^2 \leq ||x_n - x^*||^2 + 4\rho_2 K^2 \left(\rho_2 - \frac{\widetilde{\beta_2}}{2K^2}\right)||\Psi_2 x_n - \Psi_2 x^*||^2 + 4\rho_2 K^2 \left(\rho_2 - \frac{\widetilde{\gamma_2}}{2K^2}\right)||\Phi_2 x_n - \Phi_2 x^*||^2$$
$$+ 4\rho_1 K^2 \left(\rho_1 - \frac{\widetilde{\beta_1}}{2K^2}\right)||\Psi_1 z_n - \Psi_1 y^*||^2 + 4\rho_1 K^2 \left(\rho_1 - \frac{\widetilde{\gamma_1}}{2K^2}\right)||\Phi_1 z_n - \Phi_1 y^*||^2. \tag{3.27}$$

Set $u_n = Q_C v_n$, where $v_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T(t_n) y_n$. From Lemma 2.6(c), we have

$$\begin{aligned}
||u_n - x^*||^2 &= \langle v_n - x^*, j(u_n - x^*)\rangle + \langle Q_C v_n - v_n, j(Q_C v_n - x^*)\rangle \\
&\leq \langle v_n - x^*, j(u_n - x^*)\rangle \\
&= \alpha_n \langle \gamma V x_n - \mu F x^*, j(u_n - x^*)\rangle + \langle (I - \alpha_n \mu F)(T(t_n)y_n - x^*), j(u_n - x^*)\rangle \\
&\leq (1 - \alpha_n \tau)||y_n - x^*||\,||u_n - x^*|| + \alpha_n \langle \gamma V x_n - \mu F x^*, j(u_n - x^*)\rangle \\
&\leq \frac{(1 - \alpha_n \tau)}{2}\left(||y_n - x^*||^2 + ||u_n - x^*||^2\right) + \alpha_n \langle \gamma V x_n - \mu F x^*, j(u_n - x^*)\rangle \\
&\leq \frac{(1 - \alpha_n \tau)}{2}||y_n - x^*||^2 + \frac{1}{2}||u_n - x^*||^2 + \alpha_n \langle \gamma V x_n - \mu F x^*, j(u_n - x^*)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
||u_n - x^*||^2 &\leq ||\gamma_n - x^*||^2 + 2\alpha_n \langle \gamma V x_n - \mu F x^*, j(u_n - x^*)\rangle \\
&\leq ||\gamma_n - x^*||^2 + 2\alpha_n ||\gamma V x_n - \mu F x^*||\,||u_n - x^*||.
\end{aligned} \tag{3.28}$$

By the convexity of $||\cdot||^2$ and (3.28), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||\beta_n(x_n - x^*) + (1 - \beta_n)(u_n - x^*)||^2 \\
&\leq \beta_n ||x_n - x^*||^2 + (1 - \beta_n)||u_n - x_n||^2 \\
&\leq \beta_n ||x_n - x^*||^2 + (1 - \beta_n)\left\{||\gamma_n - x^*||^2 + 2\alpha_n ||\gamma V x_n - \mu F x^*||\,||u_n - x^*||\right\} \\
&= \beta_n ||x_n - x^*||^2 + (1 - \beta_n)||\gamma_n - x^*||^2 + 2\alpha_n(1 - \beta_n)||\gamma V x_n - \mu F x^*||\,||u_n - x^*||.
\end{aligned} \tag{3.29}$$

Substituting (3.27) into (3.29), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq \beta_n ||x_n - x^*||^2 + (1 - \beta_n)\left\{||x_n - x^*||^2 + 4\rho_2 K^2 \left(\rho_2 - \frac{\widetilde{\beta_2}}{2K^2}\right)||\Psi_2 x_n - \Psi_2 x^*||^2 \right. \\
&\quad + 4\rho_2 K^2 \left(\rho_2 - \frac{\widetilde{\gamma_2}}{2K^2}\right)||\Phi_2 x_n - \Phi_2 x^*||^2 + 4\rho_1 K^2 \left(\rho_1 - \frac{\widetilde{\beta_1}}{2K^2}\right)||\Psi_1 z_n - \Psi_1 y^*||^2 \\
&\quad \left. + 4\rho_1 K^2 \left(\rho_1 - \frac{\widetilde{\gamma_2}}{2K^2}\right)||\Phi_1 z_n - \Phi_1 y^*||^2\right\} + 2\alpha_n(1 - \beta_n)||\gamma V x_n - \mu F x^*||\,||u_n - x^*||\} \\
&= ||x_n - x^*||^2 + (1 - \beta_n)\left\{4\rho_2 K^2 \left(\rho_2 - \frac{\widetilde{\beta_2}}{2K^2}\right)||\Psi_2 x_n - \Psi_2 x^*||^2 \right. \\
&\quad + 4\rho_2 K^2 \left(\rho_2 - \frac{\widetilde{\gamma_2}}{2K^2}\right)||\Phi_2 x_n - \Phi_2 x^*||^2 \\
&\quad \left. 4\rho_1 K^2 \left(\rho_1 - \frac{\widetilde{\beta_1}}{2K^2}\right)||\Psi_1 z_n - \Psi_1 y^*||^2 + 4\rho_1 K^2 \left(\rho_1 - \frac{\widetilde{\gamma_1}}{2K^2}\right)||\Phi_1 z_n - \Phi_1 y^*||^2\right\} \\
&\quad + 2\alpha_n(1 - \beta_n)||\gamma V x_n - \mu F x^*||\,||u_n - x^*||,
\end{aligned}$$

which in turn implies that

$$\begin{aligned}
(1 - \beta_n)&\left\{4\rho_2 K^2 \left(\frac{\widetilde{\beta_2}}{2K^2} - \rho_2\right)||\Psi_2 x_n - \Psi_2 x^*||^2 + 4\rho_2 K^2 \left(\frac{\widetilde{\gamma_2}}{2K^2} - \rho_2\right)||\Phi_2 x_n - \Phi_2 x^*||^2 \right. \\
&\quad \left. + 4\rho_1 K^2 \left(\frac{\widetilde{\beta_1}}{2K^2} - \rho_1\right)||\Psi_1 z_n - \Psi_1 y^*||^2 + 4\rho_1 K^2 \left(\frac{\widetilde{\gamma_1}}{2K^2} - \rho_1\right)||\Phi_1 z_n - \Phi_1 y^*||^2\right\} \\
&\leq ||x_n - x^*||^2 - ||x_{n+1} - x_n||^2 + 2\alpha_n(1 - \beta_n)||\gamma V x_n - \mu F x^*||\,||u_n - x^*|| \\
&\leq \left(||x_n - x^*|| + ||x_{n+1} - x^*||\right)||x_{n+1} - x_n|| + 2\alpha_n(1 - \beta_n)||\gamma V x_n - \mu F x^*||\,||u_n - x^*||.
\end{aligned}$$

Since $\liminf_{n\to\infty}(1 - \beta_n) > 0$, $0 < \rho_i < \min\left\{\dfrac{\widetilde{\beta}_i}{2K^2}, \dfrac{\widetilde{\gamma}_i}{2K^2}\right\}$, for all $i = 1, 2$, $||x_{n+1} - x_n|| \to 0$ and $\alpha_n \to 0$, we have

$$\begin{cases} \lim_{n\to\infty}||\Psi_1 z_n - \Psi_1 y^*|| = \lim_{n\to\infty}||\Phi_1 z_n - \Phi_1 y^*|| = 0, \\ \lim_{n\to\infty}||\Psi_2 x_n - \Psi_2 x^*|| = \lim_{n\to\infty}||\Phi_2 x_n - \Phi_2 x^*|| = 0. \end{cases} \tag{3.30}$$

Let $r_1 = \sup_{n \geq 1}\{||z_n - y^*||, ||y_n - x^*||\}$. By Lemma 2.6(b) and Lemma 2.15, we have

$$\begin{aligned} ||y_n - x^*||^2 &= ||Q_C(z_n - \rho_1(\Psi_1 + \Phi_1)z_n) - Q_C(y^* - \rho_1(\Psi_1 + \Phi_1)y^*)||^2 \\ &\leq \langle z_n - \rho_1(\Psi_1 + \Phi_1)z_n - (y^* - \rho_1(\Psi_1 + \Phi_1)y^*), j(y_n - x^*)\rangle \\ &= \langle z_n - y^*, j(y_n - x^*)\rangle - \rho_1\langle(\Psi_1 + \Phi_1)z_n - (\Psi_1 + \Phi_1)y^*, j(y_n - x^*)\rangle \\ &\leq \frac{1}{2}\{||z_n - y^*||^2 + ||y_n - x^*||^2 - g_1(||z_n - y_n + x^* - y^*||)\} \\ &\quad - \rho_1\langle(\Psi_1 + \Phi_1)z_n - (\Psi_1 + \Phi_1)y^*, j(y_n - x^*)\rangle, \end{aligned}$$

which in turn implies that

$$\begin{aligned} ||y_n - x^*||^2 &\leq ||z_n - y^*||^2 - g_1(||z_n - y_n + x^* - y^*||) - 2\rho_1\langle(\Psi_1 + \Phi_1)z_n - (\Psi_1 + \Phi_1)y^*, j(y_n - x^*)\rangle \\ &\leq ||z_n - y^*||^2 - g_1(||z_n - y_n + x^* - y^*||) + 2\rho_1||(\Psi_1 + \Phi_1)z_n - (\Psi_1 + \Phi_1)y^*||||y_n - x^*||. \end{aligned} \tag{3.31}$$

Let $r_2 = \sup_{n \geq 1}\{||x_n - x^*||, ||z_n - y^*||\}$. Again, by Lemma 2.6(b) and Lemma 2.15, we have

$$\begin{aligned} ||z_n - y^*||^2 &= ||Q_C(x_n - \rho_2(\Psi_2 + \Phi_2)x_n) - Q_C(x^* - \rho_2(\Psi_2 + \Phi_2)x^*)||^2 \\ &\leq \langle x_n - \rho_2(\Psi_2 + \Phi_2)x_n - (x^* - \rho_2(\Psi_2 + \Phi_2)x^*), j(z_n - y^*)\rangle \\ &= \langle x_n - x^*, j(z_n - y^*)\rangle - \rho_2\langle(\Psi_2 + \Phi_2)x_n - (\Psi_2 + \Phi_2)x^*, j(z_n - y^*)\rangle \\ &\leq \frac{1}{2}\{||x_n - x^*||^2 + ||z_n - y^*||^2 - g_2(||x_n - z_n - (x^* - y^*)||)\} \\ &\quad - \rho_2\langle(\Psi_2 + \Phi_2)x_n - (\Psi_2 + \Phi_2)x^*, j(z_n - y^*)\rangle, \end{aligned}$$

which in turn implies that

$$\begin{aligned} ||z_n - y^*||^2 &\leq ||x_n - x^*||^2 - g_2(||x_n - z_n - (x^* - y^*)||) - 2\rho_2\langle(\Psi_2 + \Phi_2)x_n - (\Psi_2 + \Phi_2)x^*, j(z_n - y^*)\rangle \\ &\leq ||x_n - x^*||^2 - g_2(||x_n - z_n - (x^* - y^*)||) + 2\rho_2||(\Psi_2 + \Phi_2)x_n - (\Psi_2 + \Phi_2)x^*||||z_n - y^*||. \end{aligned} \tag{3.32}$$

Substituting (3.32) into (3.31), we obtain

$$\begin{aligned} ||y_n - x^*||^2 &\leq ||x_n - x^*||^2 - g_1(||z_n - y_n + x^* - y^*||) - g_2(||x_n - z_n - (x^* - y^*)||) \\ &\quad + 2\rho_1||(\Psi_1 + \Phi_1)z_n - (\Psi_1 + \Phi_1)y^*||||y_n - x^*|| + 2\rho_2||(\Psi_2 + \Phi_2)x_n \\ &\quad - (\Psi_2 + \Phi_2)x^*||||z_n - y^*||. \end{aligned} \tag{3.33}$$

And, then substituting (3.33) into (3.29), we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq \beta_n||x_n - x^*||^2 + (1 - \beta_n)\{||x_n - x^*||^2 - g_1(||z_n - y_n + x^* - y^*||) \\ &\quad - g_2(||x_n - z_n - (x^* - y^*)||) \\ &\quad + 2\rho_1||(\Psi_1 + \Phi_1)z_n - (\Psi_1 + \Phi_1)y^*||||y_n - x^*|| + 2\rho_2||(\Psi_2 + \Phi_2)x_n \\ &\quad - (\Psi_2 + \Phi_2)x^*||||z_n - y^*||\} \\ &\quad + 2\alpha_n(1 - \beta_n)||\gamma V x_n - \mu F x^*||||u_n - x^*|| \\ &\leq ||x_n - x^*||^2 - (1 - \beta_n)\{g_1(||z_n - y_n + x^* - y^*||) - g_2(||x_n - z_n - (x^* - y^*)||)\} \\ &\quad + 2\rho_1||(\Psi_1 + \Phi_1)z_n - (\Psi_1 + \Phi_1)y^*||||y_n - x^*|| + 2\rho_2||(\Psi_2 + \Phi_2)x_n \\ &\quad - (\Psi_2 + \Phi_2)x^*||||z_n - y^*|| \\ &\quad + 2\alpha_n(1 - \beta_n)||\gamma V x_n - \mu F x^*||||u_n - x^*||, \end{aligned}$$

which in turn implies that

$$
\begin{aligned}
(1 - \beta_n) \{ g_1 &(||z_n - y_n + x^* - y^*||) + g_2 (||x_n - z_n - (x^* - y^*)||) \} \\
&\leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + 2\rho_1 ||(\Psi_1 + \Phi_1) z_n - (\Psi_1 + \Phi_1)|| ||y_n - x^*|| \\
&\quad + 2\rho_2 ||(\Psi_2 + \Phi_2) x_n - (\Psi_2 + \Phi_2) x^*|| ||z_n - y^*|| + 2\alpha_n (1 - \beta_n) ||\gamma V x_n - \mu F x^*|| ||u_n - x^*|| \\
&\leq (||x_n - x^*|| + ||x_{n+1} - x^*||) ||x_{n+1} - x_n|| + 2\rho_1 ||(\Psi_1 + \Phi_1) z_n - (\Psi_1 + \Phi_1) y^*|| ||y_n - x^*|| \\
&\quad + 2\rho_2 ||(\Psi_2 + \Phi_2) x_n - (\Psi_2 + \Phi_2) x^*|| ||z_n - y^*|| + 2\alpha_n (1 - \beta_n) ||\gamma V x_n - \mu F x^*|| ||u_n - x^*||.
\end{aligned}
$$

Since $\liminf_{n\to\infty}(1 - \beta_n) > 0$, $||x_{n+1} - x_n|| \to 0$, $\alpha_n \to 0$ and (3.30), we have

$$
\lim_{n\to\infty} g_1 (||z_n - y_n + x^* - y^*||) = \lim_{n\to\infty} g_2 (||x_n - z_n - (x^* - y^*)||) = 0.
$$

It follows from the properties of g_1 and g_2 that

$$
\lim_{n\to\infty} ||z_n - y_n + x^* - y^*|| = \lim_{n\to\infty} ||x_n - z_n - (x^* - y^*)|| = 0.
$$

Consequently, we have

$$
||x_n - y_n|| \leq ||x_n - z_n - (x^* - y^*)|| + ||z_n - y_n + x^* - y^*|| \to 0 \text{ as } n \to \infty. \quad (3.34)
$$

On the other hand, we observe that

$$
\begin{aligned}
||x_n - T(t_n) x_n|| &\leq ||x_n - T(t_n) y_n|| + ||T(t_n) y_n - T(t_n) x_n|| \\
&\leq ||x_n - T(t_n) y_n|| + ||y_n - x_n||.
\end{aligned}
$$

From (3.24) and (3.34), we obtain that

$$
\lim_{n\to\infty} ||x_n - T(t_n) x_n|| = 0. \quad (3.35)
$$

For all $h > 0$, we note that

$$
\begin{aligned}
||x_n - T(h) x_n|| &\leq ||x_n - T(t_n) x_n|| + ||T(t_n) x_n - T(h) T(t_n) x_n|| + ||T(h) T(t_n) x_n - T(h) x_n|| \\
&\leq 2||x_n - T(t_n) x_n|| + ||T(t_n) x_n - T(h) T(t_n) x_n||.
\end{aligned}
$$

Since $\{T(h): h > 0\}$ is a u.a.r. nonexpansive semigroup, it follows from (3.20) and (3.35) that

$$
\lim_{n\to\infty} ||x_n - T(h) x_n|| = 0. \quad (3.36)
$$

Since $\{T(h): h > 0\}$ is a u.a.r. nonexpansive semigroup, by Lemma 2.13, for each $x \in C$, there exists a sequence $\{T(t_k): t_k > 0, k \in \mathbb{N}\} \{T(h): h > 0\}$ such that $\{T(t_k)x\}$ converges strongly to some point in $\text{Fix}(\mathcal{S})$, where $t_k \to \infty$ as $k \to \infty$. Define a mapping $T : C \to C$ by

$$
Tx = \lim_{k\to\infty} T(t_k) x, \quad \forall x \in C.
$$

By [25, Remark 3.4], we see that the mapping T is nonexpansive such that $\text{Fix}(T) = \text{Fix}(\mathcal{S})$. From (3.36), we obtain that

$$
\begin{aligned}
\lim_{n\to\infty} ||x_n - Tx_n|| &= \lim_{n\to\infty} \lim_{k\to\infty} ||x_n - T(t_k) x_n|| \\
&= \lim_{k\to\infty} \lim_{n\to\infty} ||x_n - T(t_k) x_n|| = 0.
\end{aligned} \quad (3.37)
$$

Define a mapping $W : C \to C$ by

$$
Wx = \delta Tx + (1 - \delta) Gx, \quad \forall x \in C,
$$

where δ is a constant in $(0, 1)$. By Lemma 2.11, we see that the mapping W is nonexpansive such that

$$\text{Fix}(W) = \text{Fix}(T) \cap \text{Fix}(G)$$

Notice that

$$
\begin{aligned}
||x_n - Wx_n|| &\leq ||\delta(x_n - Tx_n) + (1 - \delta)(x_n - Gx_n)|| \\
&= ||\delta(x_n - Tx_n) + (1 - \delta)(x_n - y_n)|| \\
&\leq \delta||x_n - Tx_n|| + (1 - \delta)||x_n - y_n||.
\end{aligned}
$$

From (3.34) and (3.37), we obtain that

$$\lim_{n \to \infty}||x_n - Wx_n|| = 0. \tag{3.38}$$

Next, we show that

$$\limsup_{n \to \infty}\langle \gamma V\widehat{x} - \mu F\widehat{x}, j(u_n - \widehat{x})\rangle \leq 0,$$

where $\widehat{x} = \lim_{t \to 0}x_t$ and x_t is the unique fixed point of the contraction mapping $T_t : C \to C$ given by

$$T_t x = Q_C[t\gamma Vx + (I - t\mu F)Wx], \quad \forall x \in C \text{ and } t \in \left(0, \min\left\{1, \frac{1}{\tau}\right\}\right).$$

By Lemma 3.1, we have $\widehat{x} \in \text{Fix}(W) = \Omega$, which solves the variational inequality

$$\langle(\mu F - \gamma V)\widehat{x}, j(\widehat{x} - v)\rangle \leq 0, \quad \forall v \in \Omega.$$

By (3.21) and Lemma 3.2, we obtain that

$$
\begin{aligned}
\limsup_{n \to \infty}\langle \gamma V\widehat{x} - \mu F\widehat{x}, j(u_n - \widehat{x})\rangle &= \limsup_{n \to \infty}\langle \gamma V\widehat{x} - \mu F\widehat{x}, j(x_n - \widehat{x})\rangle \\
&\leq 0.
\end{aligned}
\tag{3.39}
$$

Finally, we show that $x_n \to \widehat{x}$ as $n \to \infty$. Notice that $u_n = Q_C v_n$, where $v_n = \alpha_n \gamma V x_n + (I - \alpha_n\mu F)T(t_n)y_n$. Then, from Lemma 2.6(c), we have

$$
\begin{aligned}
||u_n - \widehat{x}||^2 &= \langle v_n - \widehat{x}, j(u_n - \widehat{x})\rangle + \langle Q_C v_n - v_n, j(Q_C v_n - \widehat{x})\rangle \\
&\leq \langle v_n - \widehat{x}, j(u_n - \widehat{x})\rangle \\
&= \alpha_n\gamma\langle Vx_n - V\widehat{x}, j(u_n - \widehat{x})\rangle + \alpha_n\langle \gamma V\widehat{x} - \mu F\widehat{x}, j(u_n - \widehat{x})\rangle \\
&\quad + \langle(I - \alpha_n\mu F)(T(t_n)y_n - \widehat{x}), j(u_n - \widehat{x})\rangle \\
&\leq (1 - (\tau - \gamma L)\alpha_n(1 - \beta_n))||x_n - \widehat{x}||^2 + \alpha_n\langle \gamma V\widehat{x} - \mu F\widehat{x}, j(u_n - \widehat{x})\rangle.
\end{aligned}
\tag{3.40}
$$

It follows from (3.40) that

$$
\begin{aligned}
||x_{n+1} - \widehat{x}||^2 &\leq \beta_n||x_n - \widehat{x}||^2 + (1 - \beta_n)||u_n - \widehat{x}||^2 \\
&\leq \beta_n||x_n - \widehat{x}||^2 + (1 - \beta_n)\{(1 - (\tau - \gamma L)\alpha_n(1 - \beta_n))||x_n - \widehat{x}||^2 \\
&\quad + \alpha_n\langle \gamma V\widehat{x} - \mu F\widehat{x}, j(u_n - \widehat{x})\rangle\} \\
&= (1 - (\tau - \gamma L)\alpha_n(1 - \beta_n))||x_n - \widehat{x}||^2 + \alpha_n(1 - \beta_n)\langle \gamma V\widehat{x} - \mu F\widehat{x}, j(u_n - \widehat{x})\rangle.
\end{aligned}
\tag{3.41}
$$

Put $\sigma_n := (\tau - \gamma L)\alpha_n(1 - \beta_n)$ and $\delta_n := \alpha_n(1 - \beta_n)\langle \gamma V\widehat{x} - \mu F\widehat{x}, j(u_n - \widehat{x})\rangle$. Then (3.41) re-duces to formula

$$||x_{n+1} - \widehat{x}||^2 \leq (1 - \sigma_n)||x_n - \widehat{x}||^2 + \delta_n.$$

It is easily seen that $\sum_{n=1}^{\infty} \sigma_n = \infty$ and (using 3.39)

$$\limsup_{n \to \infty} \frac{\delta_n}{\sigma_n} = \frac{1}{\tau - \gamma L} \limsup_{n \to \infty} \langle \gamma V \widehat{x} - \mu F \widehat{x}, j(u_n - \widehat{x}) \rangle \leq 0.$$

Hence, by Lemma 2.16, we conclude that $x_n \to \widehat{x}$ as $n \to \infty$. This completes the proof.

Remark 3.4. Note that Lemma 2.17 and Lemma 2.18 play an important role in the proof of Theorem 3.3. These are proved in the framework of the more general uniformly convex and 2-uniformly smooth Banach space. Lemma 2.17 is quite similar to the result of Yamada [10] which is obtained in a Hilbert space but we extended that result to a Banach space.

Remark 3.5. Theorem 3.3 extends the main result of Yao et al. [7] in the following ways:

(i) A general system of variational inequalities (1.12) containing two inverse-strongly accre-tive mappings are extends to a general system of nonlinear variational inequalities (1.15) containing perturbed mappings.

(ii) Theorem 3.3 for finding an element $\widehat{x} \in \text{Fix}(\mathcal{S}) \cap \text{Fix}(G)$ (G is defined as in Lemma 2.19) is more general the one of finding elements of $\widehat{x} \in VI(C, \Psi_1) \cap VI(C, \Psi_2)$ of Yao et al. [7].

Furthermore, our method of the proof is very different from that in [7, Theorem 3.7] because it can be applied to solving the problem of finding a common element of the set of common fixed points of a one-parameter nonexpansive semigroup and the set of solutions of a general system of nonlinear variational inequalities containing perturbed mappings.

From Theorem 3.3, Lemma 2.4 and Example 2.5, we have the following result.

Corollary 3.6. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth and uniformly convex Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $F : C \to X$ be a κ-Lipschitizian and η-strongly accretive operator with constants κ, $\eta > 0$, $V : C \to C$ be an L-Lipschitzian mapping with a constant $L \geq 0$. Let $0 < \mu < \frac{\eta}{\kappa^2 K^2}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \mu \kappa^2 K^2)$. Let $\mathcal{S} = \{T(h) : h > 0\}$ be a u.a.r. nonexpansive semigroup from C into itself such that $\text{Fix}(\mathcal{S}) := \cap_{h>0} \text{Fix}(T(h)) \neq \emptyset$ and least there exists a T(h) which is demicompact. Let $\Psi_i : C \to X$ (i = 1, 2) be $\widetilde{\beta}_i$-inverse-strongly accretive and $\Phi_i : C \to X$ (i = 1, 2) be $\widetilde{\gamma}_i$-inverse-strongly accretive. Assume that $\Omega := \text{Fix}(\mathcal{S}) \cap \text{Fix}(G) \neq \emptyset$, where G is defined as in Lemma 2.19. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} z_n = Q_C (x_n - \rho_2 (\Psi_2 + \Phi_2) x_n), \\ y_n = Q_C (z_n - \rho_1 (\Psi_1 + \Phi_1) z_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) Q_C \left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) \frac{1}{t_n} \int_0^{t_n} T(s) y_n \mathrm{d}s \right], \quad \forall n \geq 1, \end{cases} \quad (3.42)$$

where $\rho_i \in \left(0, \min \left\{ \frac{\widetilde{\beta}_i}{2K^2}, \frac{\widetilde{\gamma}_i}{2K^2} \right\} \right)$ for all i = 1, 2. Suppose that $\{t_n\}$ is a positive real divergent sequence such that $\lim_{n \to \infty} \frac{t_n}{t_{n+1}} = 1$, and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0, 1] satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \lim \inf_{n\to\infty} \beta_n \le \lim \sup_{n\to\infty} \beta_n < 1$.

Then, the sequence $\{x_n\}$ *defined by* (3.42) *converges strongly to* $\widehat{x} \in \Omega$ *as* $n \to \infty$, *which* $\widetilde{x}$ *is the unique solution of the variational inequality*

$$\langle (\mu F - \gamma V)\, \widetilde{x}, j\, (\widetilde{x} - v) \rangle \le 0, \quad \forall v \in \Omega, \tag{3.43}$$

and $(\widehat{x}, \widehat{\gamma})$ *is the solution of the problem* (1.15), *where* $\widehat{\gamma} = Q_C\, (\widehat{x} - \rho_2\, (\Psi_2 + \Phi_2)\, \widehat{x})$.

Authors' contributions
PS and PK conceived the idea, designed the research and wrote the article; PS conducted the research. Both authors have read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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