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Existence and uniqueness of solutions for a class of integral equations by common fixed point theorems in IFMT-spaces

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Abstract

In this paper, our aim is to address the existence and uniqueness of solutions for a class of integral equations in IFMT-space. Therefore, we introduce the concept of IFMT-spaces and prove a common fixed point theorem in a complete IFMT-space; next we study an application.

MSC: 54E40; 54E35; 54H25

Keywords: integral equations; nonlinear IF contractive mapping; complete IFMT-space; fixed point theorem

1 Introduction and preliminaries

First of all, we would like to introduce the concept of IFMT-space, which is a non-trivial generalization of IFM-space introduced by Park [1] and Saadati and Park [2] and Saadati *et al.* [3]; also we use results from [4–8].

We say the pair $(L^*, \leq_{L^*})$ is a complete lattice whenever L^* is a non-empty set and we have the operation $\leq_{L^*}$ defined by

$$L^* = \{(a,b) : (a,b) \in [0,1] \times [0,1] \text{ and } a + b \leq 1\},$$

$(a,b) \leq_{L^*} (c,d) \Longleftrightarrow a \leq c$, and $b \geq d$, for each $(a,b),(c,d) \in L^*$.

Definition 1.1 ([9]) An IF set $\mathcal{F}_{\alpha,\beta}$ in a universe U is an object $\mathcal{F}_{\alpha,\beta} = \{(\alpha_{\mathcal{F}}(u), \beta_{\mathcal{F}}(u)) | u \in U\}$, in which, for all $u \in U$, $\alpha_{\mathcal{F}}(u) \in [0,1]$, and $\beta_{\mathcal{F}}(u) \in [0,1]$ are said the membership degree and the non-membership degree, respectively, of u in $\mathcal{F}_{\alpha,\beta}$, and furthermore they satisfy $\alpha_{\mathcal{F}}(u) + \beta_{\mathcal{F}}(u) \leq 1$.

We consider $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$ as its units.

Definition 1.2 ([4]) The mapping $\mathcal{T} : L^* \times L^* \longrightarrow L^*$ satisfying the following conditions:

$(\forall a \in L^*)\ (\mathcal{T}(a, 1_{L^*}) = a)$,
$(\forall (a,b) \in L^* \times L^*)\ (\mathcal{T}(a,b) = \mathcal{T}(b,a))$,
$(\forall (a,b,c) \in L^* \times L^* \times L^*)\ (\mathcal{T}(a, \mathcal{T}(b,c)) = \mathcal{T}(\mathcal{T}(a,b),c))$,
$(\forall (a,a',b,b') \in L^* \times L^* \times L^* \times L^*)\ (a \leq_{L^*} a' \text{ and } b \leq_{L^*} b' \Longrightarrow \mathcal{T}(a,b) \leq_{L^*} \mathcal{T}(a',b'))$.

is said to be a triangular norm (t-norm) on L^*.

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$\mathcal{T}$ is said to be a *continuous* t-norm if the triple $(L^*, \leq_{L^*}, \mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*}.

Definition 1.3 ([4]) $\mathcal{T}$ on L^* is called continuous *t-representable* if and only if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0,1]$ such that, for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$,

$$\mathcal{T}(a,b) = (a_1 * b_1, a_2 \diamond b_2).$$

For example, $\mathcal{T}(a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* is a continuous t-representable.

Definition 1.4 The decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$ is said a negator on L^*. We say $\mathcal{N}$ is an involutive negator if $\mathcal{N}(\mathcal{N}(a)) = a$, for all $a \in L^*$. The decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$ is said to be a negator on $[0,1]$. The standard negator on $[0,1]$ is defined, for all $a \in [0,1]$, by $N_s(a) = 1 - a$, denoted by N_s. We show $(N_s(a), a) = \mathcal{N}_s(a)$.

Definition 1.5 If for given $\alpha \in (0,1)$ there is $\beta \in (0,1)$ such that

$$\mathcal{T}^m\big(\mathcal{N}_s(\beta), \dots, \mathcal{N}_s(\beta)\big) >_{L^*} \mathcal{N}_s(\alpha), \quad m \in \mathbf{N},$$

then $\mathcal{T}$ is a *H-type* t-norm.

A typical example of such t-norms is

$$\wedge(a,b) = \big(\mathrm{Min}(a_1, b_1), \mathrm{Max}(a_2, b_2)\big),$$

for every $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^*.

Definition 1.6 The tuble $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an *IFMT-space* if X is an (non-empty) set, $\mathcal{T}$ is a continuous t-representable, and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times [0, +\infty) \to L^*$ (in which M, N are fuzzy sets from $X^2 \times [0, +\infty)$ to $[0,1]$ such that $M(x,y,t) + N(x,y,t) \leq 1$ for all $x, y \in X$ and $t > 0$) satisfying the following conditions for every $x, y, z \in X$ and $t, s > 0$:

(a) $\mathcal{M}_{M,N}(x,y,t) >_L 0_{L^*}$;
(b) $\mathcal{M}_{M,N}(x,y,t) = \mathcal{M}_{M,N}(y,x,t) = 1_{L^*}$ iff $x = y$;
(c) $\mathcal{M}_{M,N}(x,y,t) = \mathcal{M}_{M,N}(y,x,t)$ for each $x, y \in X$;
(d) $\mathcal{M}_{M,N}(x,y,K(t+s)) \geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x,z,t), \mathcal{M}_{M,N}(z,y,s))$ for some constant $K \geq 1$;
(e) $\mathcal{M}_{M,N}(x,y,\cdot) : [0,\infty) \longrightarrow L^*$ is continuous.

Also $\mathcal{M}_{M,N}$ is said an *IFMT*. Note that for an IFMT-space

$$\mathcal{M}_{M,N}(x,y,t) = \big(M(x,y,t), N(x,y,t)\big).$$

$(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called a *Menger* IFMT-space if

$$\lim_{t\to\infty} \mathcal{M}_{M,N}(x,y,t) = \lim_{t\to\infty} \mathcal{M}_{M,N}(y,x,t) = 1_{L^*}.$$

Remark 1.7 The space of all real functions $\alpha(x)$, $x \in [0,1]$ such that $\int_0^1 |\alpha(x)|^q\,dx < \infty$, denoted by L_q $(0 < q < 1)$, is a metric type space. Consider

$$d(\alpha,\beta) = \left(\int_0^1 \left|\alpha(x) - \beta(x)\right|^q dx\right)^{\frac{1}{q}},$$

for each $\alpha, \beta \in L_q$. Then d is a metric type space with $K = 2^{\frac{1}{q}}$.

Example 1.8 We consider the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |\alpha(x)|^q\,dx < \infty$, where $q > 0$ is a real number denoted by $\mathfrak{M}$. Consider

$$\mathcal{M}_{M,N}(x,y,t) = \begin{cases} 0_{L^*} & \text{if } t \leq 0, \\ \left(\frac{t}{t+(\int_0^1 |\alpha(x)-\beta(x)|^q\,dx)^{\frac{1}{q}}}, \frac{(\int_0^1 |\alpha(x)-\beta(x)|^q\,dx)^{\frac{1}{q}}}{t+(\int_0^1 |\alpha(x)-\beta(x)|^q\,dx)^{\frac{1}{q}}}\right) & \text{if } t > 0. \end{cases}$$

So from Remark 1.7, we have $(M, \mathcal{M}_{M,N}, \wedge)$ is IFMT-space with $K = 2^{\frac{1}{q}}$.

Definition 1.9 Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Menger IFMT-space.

(1) A sequence $\{x_n\}_n$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda \in 0$, there exists a positive integer N such that $\mathcal{M}_{M,N}(x_n, x, \epsilon) > 1 - \lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}_n$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda L^* - \{0_{L^*}\}$, there exists a positive integer N such that $\mathcal{M}_{M,N}(x_n, x_m, \epsilon) >_L \mathcal{N}(\lambda)$ whenever $n, m \geq N$.
(3) A Menger IFMT-space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Remark 1.10 Khamsi and Kreinovich [10] proved, if $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is a IFMT-space and $\{u_n\}$ and $\{v_n\}$ are sequences such that $u_n \to u$ and $v_n \to v$, then

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(u_n, v_n, t) = \mathcal{M}_{M,N}(u, v, t).$$

Remark 1.11 Let for each $\sigma \in L^* - \{0_{L^*}, 1_{L^*}\}$ there exists a $\varsigma \in L^* - \{0_{L^*}, 1_{L^*}\}$ (which does not depend on n) with

$$\mathcal{T}^{n-1}\big(\mathcal{N}(\varsigma), \ldots, \mathcal{N}(\varsigma)\big) >_L \mathcal{N}(\sigma) \quad \text{for each } n \in \{1, 2, \ldots\}. \tag{1}$$

Lemma 1.12 ([11]) *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Menger IFMT-space. If we define $E_{\varsigma,\mathcal{M}_{M,N}} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\varsigma,\mathcal{M}_{M,N}}(x,y) = \inf\big\{t > 0 : \mathcal{M}_{M,N}(x,y,t) >_L \mathcal{N}(\varsigma)\big\}$$

for each $\varsigma \in L^* - \{0_{L^*}, 1_{L^*}\}$ and $x, y \in X$, then we have the following:

(1) *For any $\sigma \in L^* - \{0_{L^*}, 1_{L^*}\}$, there exists a $\varsigma \in L^* - \{0_{L^*}, 1_{L^*}\}$ such that*

$$E_{\mu,\mathcal{M}_{M,N}}(x_1, x_k) \leq K E_{\varsigma,\mathcal{M}_{M,N}}(x_1, x_2) + K^2 E_{\varsigma,\mathcal{M}_{M,N}}(x_2, x_3) + \cdots + K^{n-1} E_{\varsigma,\mathcal{M}_{M,N}}(x_{k-1}, x_k)$$

for any $x_1, \ldots, x_k \in X$.

(2) *For each sequence* $\{x_n\}$ *in* X, *we have* $\mathcal{M}_{M,N}(x_n, x, t) \longrightarrow 1_{L^*}$ *if and only if* $E_{\varsigma,\mathcal{M}_{M,N}}(x_n, x) \to 0$. *Also the sequence* $\{x_n\}$ *is Cauchy w.r.t.* $\mathcal{M}_{M,N}$ *if and only if it is Cauchy with* $E_{\varsigma,\mathcal{M}_{M,N}}$.

2 Common fixed point theorems

In this section we study some common fixed point theorems in Menger IFMT-spaces, ones can find similar results in others spaces at [12–19].

Definition 2.1 Let f and g be mappings from a Menger IFMT-space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ into itself. The mappings f and g are called weakly commuting if

$$\mathcal{M}_{M,N}(fgx, gfx, t) \geq_L \mathcal{M}_{M,N}(fx, gx, t)$$

for each x in X and $t > 0$.

Now we assume that Φ is the set of all functions

$$\phi : [0, \infty) \longrightarrow [0, \infty)$$

which satisfy $\lim_{n\to\infty} \phi^n(t) = 0$ for $t > 0$ and are onto and strictly increasing. Also, we denote by $\phi^n(t)$ the nth iterative function of $\phi(t)$.

Remark 2.2 Note that $\phi \in \Phi$ implies that $\phi(t) < t$ for $t > 0$. Consider $t_0 > 0$ with $t_0 \leq \phi(t_0)$. Since ϕ is a nondecreasing function we get $t_0 \leq \phi^n(t_0)$ for every $n \in \{1, 2, \ldots\}$, which is a contradiction. Also $\phi(0) = 0$.

Lemma 2.3 ([11]) *If a Menger IFMT-space* $(X, \mathcal{M}_{M,N}, \mathcal{T})$ *obeys the condition*

$$\mathcal{M}_{M,N}(x, y, t) = C, \quad \text{for all } t > 0,$$

then we get $C = 1_{L^*}$ *and* $x = y$.

Theorem 2.4 *Consider the complete Menger IFMT-space* $(X, \mathcal{M}_{M,N}, \mathcal{T})$. *Assume that* f *and* g *are weakly commuting self-mappings of* X *such that*:

(a) $f(X) \subseteq g(X)$;
(b) f *or* g *is continuous*;
(c) $\mathcal{M}_{M,N}(fx, fy, \phi(t)) \geq_L \mathcal{M}_{M,N}(gx, gy, t)$ *in which* $\phi \in \Phi$.
(i) *Now let* (1) *hold and let there exist a* $x_0 \in X$ *with*

$$E_{\mathcal{M}_{M,N}}(gx_0, fx_0) = \sup\bigl\{E_{\gamma,\mathcal{M}_{M,N}}(gx_0, fx_0) : \gamma \in L^* - \{0_{L^*}, 1_{L^*}\}\bigr\} < \infty,$$

therefore f *and* g *have a common fixed point which is unique.*

Proof (i) Select $x_0 \in X$ with $E_{\mathcal{M}_{M,N}}(gx_0, fx_0) < \infty$. Select $x_1 \in X$ with $fx_0 = gx_1$. Now select x_{n+1} such that $fx_n = gx_{n+1}$. Now $\mathcal{M}_{M,N}(fx_n, fx_{n+1}, \phi^{n+1}(t)) \geq_L \mathcal{M}_{M,N}(gx_n, gx_{n+1}, \phi^n(t)) = \mathcal{M}_{M,N}(fx_{n-1}, fx_n, \phi^n(t)) \geq_L \cdots \geq \mathcal{M}_{M,N}(gx_0, gx_1, t)$.

We have for each $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$ (see Lemma 1.9 of [11])

$$\begin{aligned} E_{\lambda,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) &= \inf\{\phi^{n+1}(t) > 0 : \mathcal{M}_{M,N}(fx_n, fx_{n+1}, \phi^{n+1}(t)) >_L \mathcal{N}(\lambda)\} \\ &\leq \inf\{\phi^{n+1}(t) > 0 : \mathcal{M}_{M,N}(gx_0, fx_0, t) >_L \mathcal{N}(\lambda)\} \\ &\leq \phi^{n+1}(\inf\{t > 0 : \mathcal{M}_{M,N}(gx_0, fx_0, t) >_L \mathcal{N}(\lambda)\}) \\ &= \phi^{n+1}(E_{\lambda,\mathcal{M}_{M,N}}(gx_0, fx_0)) \\ &\leq \phi^{n+1}(E_{\mathcal{M}_{M,N}}(gx_0, fx_0)). \end{aligned}$$

Thus $E_{\lambda,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) \leq \phi^{n+1}(E_{\mathcal{M}_{M,N}}(gx_0, fx_0))$ for each $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$ and so

$$E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) \leq \phi^{n+1}(E_{\mathcal{M}_{M,N}}(gx_0, fx_0)).$$

Let $\epsilon > 0$. Select $n \in \{1, 2, \dots\}$; therefore $E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) < \frac{\epsilon - \phi(\epsilon)}{K}$. For $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$ there exists a $\mu \in L^* - \{0_{L^*}, 1_{L^*}\}$ with

$$\begin{aligned} E_{\lambda,\mathcal{M}_{M,N}}(fx_n, fx_{n+2}) &\leq KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + KE_{\mu,\mathcal{M}_{M,N}}(fx_{n+1}, fx_{n+2}) \\ &\leq KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + \phi(KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1})) \\ &\leq KE_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + \phi(KE_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1})) \\ &\leq K\frac{\epsilon - \phi(\epsilon)}{K} + \phi\left(K\frac{\epsilon - \phi(\epsilon)}{K}\right) \\ &\leq \epsilon. \end{aligned}$$

We can continue this process for every $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$; then

$$E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+2}) \leq \epsilon.$$

For $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$ there exists a $\mu \in L^* - \{0_{L^*}, 1_{L^*}\}$ with

$$\begin{aligned} E_{\lambda,\mathcal{M}_{M,N}}(fx_n, x_{n+3}) &\leq KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + KE_{\mu,\mathcal{M}_{M,N}}(fx_{n+1}, fx_{n+3}) \\ &\leq KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + \phi(KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+2})) \\ &\leq KE_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + \phi(KE_{\mathcal{M}_{M,N}}(fx_n, fx_{n+2})) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon, \end{aligned}$$

from $\mathcal{M}_{M,N}(fx_{n+1}, fx_{n+3}, \phi(t)) \geq_L \mathcal{M}_{M,N}(gx_{n+1}, gx_{n+3}, t) = \mathcal{M}_{M,N}(fx_n, fx_{n+2}, t)$ we have $E_{\lambda,\mathcal{M}_{M,N}}(fx_{n+1}, fx_{n+3}) \leq \phi(E_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+2}))$, which implies that

$$E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+3}) \leq \epsilon.$$

By using induction

$$E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+k}) \leq \epsilon \quad \text{for } k \in \{1, 2, \dots\},$$

and we conclude that $\{fx_n\}_n$ is a Cauchy sequence and by the completeness of X, $\{fx_n\}_n$ converges to a point named z in X. Also $\{gx_n\}_n$ converges to z. Now we assume that the mapping f is continuous. Then $\lim_n ffx_n = fz$ and $\lim_n fgx_n = fz$. Also, since f and g are weakly commuting,

$$\mathcal{M}_{M,N}(fgx_n, gfx_n, t) \geq_L \mathcal{M}_{M,N}(fx_n, gx_n, t).$$

Take $n \to \infty$ in the above inequality and we get $\lim_n gfx_n = fz$, by the continuity of $\mathcal{M}$. Now, we show that $z = fz$. Assume that $z \neq fz$. From (c) for each $t > 0$ we have

$$\mathcal{M}_{M,N}\big(fx_n, ffx_n, \phi^{k+1}(t)\big) \geq_L \mathcal{M}_{M,N}\big(gx_n, gfx_n, \phi^k(t)\big), \quad k \in \mathbb{N}.$$

Suppose that $n \to \infty$ in the above inequality; we get

$$\mathcal{M}_{M,N}\big(z, fz, \phi^{k+1}(t)\big) \geq_L \mathcal{M}_{M,N}\big(z, fz, \phi^k(t)\big).$$

Furthermore we have

$$\mathcal{M}_{M,N}\big(z, fz, \phi^k(t)\big) \geq_L \mathcal{M}_{M,N}\big(z, fz, \phi^{k-1}(t)\big)$$

and

$$\mathcal{M}_{M,N}\big(z, fz, \phi(t)\big) \geq_L \mathcal{M}_{M,N}(z, fz, t).$$

Also

$$\mathcal{M}_{M,N}\big(z, fz, \phi^{k+1}(t)\big) \geq_L \mathcal{M}_{M,N}(z, fz, t).$$

Next, we have (see Remark 2.2)

$$\mathcal{M}_{M,N}\big(z, fz, \phi^{k+1}(t)\big) \leq_L \mathcal{M}_{M,N}(z, fz, t).$$

Then $\mathcal{M}_{M,N}(z, fz, t) = C$ and from Lemma 2.3, we conclude that $z = fz$. By assumption we have $f(X) \subseteq g(X)$; then there exists a z_1 in X such that $z = fz = gz_1$. Now,

$$\mathcal{M}_{M,N}(ffx_n, fz_1, t) \geq_L \mathcal{M}_{M,N}\big(gfx_n, gz_1, \phi^{-1}(t)\big).$$

Take $n \to \infty$; we get

$$\mathcal{M}_{M,N}(fz, fz_1, t) \geq_L \mathcal{M}_{M,N}\big(fz, gz_1, \phi^{-1}(t)\big) = 1_{L^*},$$

then $fz = fz_1$, *i.e.*, $z = fz = fz_1 = gz_1$. Also for each $t > 0$ we get

$$\mathcal{M}_{M,N}(fz, gz, t) = \mathcal{M}_{M,N}(fgz_1, gfz_1, t) \geq_L \mathcal{M}_{M,N}(fz_1, gz_1, t) = \varepsilon_0(t)$$

since f and g are weakly commuting, from which we can conclude that $fz = gz$. This implies that z is a common fixed point of f and g.

Now we prove the uniqueness. Assume that $z' \neq z$ is another common fixed point of f and g. Now, for each $t > 0$ and $n \in \mathbb{N}$, we have

$$\mathcal{M}_{M,N}(z, z', \phi^{n+1}(t)) = \mathcal{M}_{M,N}(fz, fz', \phi^{n+1}(t)) \geq_L F_{gz,gz'}(\phi^n(t)) = F_{z,z'}(\phi^n(t)).$$

Also of course we have

$$\mathcal{M}_{M,N}(z, z', \phi^n(t)) \geq_L \mathcal{M}_{M,N}(z, z', \phi^{n-1}(t))$$

and

$$\mathcal{M}_{M,N}(z, z', \phi^n(t)) \geq_L \mathcal{M}_{M,N}(z, z', t).$$

As a result

$$\mathcal{M}_{M,N}(z, z', \phi^{n+1}(t)) \geq_L \mathcal{M}_{M,N}(z, z', t).$$

On the other hand we have

$$\mathcal{M}_{M,N}(z, z', t) \leq_L \mathcal{M}_{M,N}(z, z', \phi^{n+1}(t)).$$

Then $\mathcal{M}_{M,N}(z, z', t) = C$, see Lemma 2.3, implies that $z = z'$, which is contradiction. Then z is the unique common fixed point of f and g. □

3 The existence and uniqueness of solutions for a class of integral equations

Assume that $X = C([1,3], (-\infty, 2.1443888))$ and

$$\mathcal{M}_{M,N}(x, y, t) = \begin{cases} 0 & \text{if } t \leq 0, \\ (\inf_{\ell \in [1,3]} \frac{t}{t + (x(\ell) - y(\ell))^2}, \sup_{\ell \in [1,3]} \frac{(x(\ell) - y(\ell))^2}{t + (x(\ell) - y(\ell))^2}) & \text{if } t > 0, \end{cases}$$

for $x, y \in X$, then $(M, \mathcal{M}_{M,N}, \wedge)$ is a complete IFTM-space with $K = 2$.

We consider the mapping $T : X \to X$ by

$$T(x(\ell)) = 4 + \int_1^\ell (x(u) - u^2) e^{1-u} \, du.$$

Put $g(x) = T(x)$ and $f(x) = T^2(x)$. Since $fg = gf$, f and g are (weakly) commuting. Now, for $x, y \in X$ and $t > 0$,

$$\begin{aligned}
&\mathcal{M}_{M,N}(fx, fy, t) \\
&\quad = \mathcal{M}_{M,N}(T(Tx(\ell)), T(Ty(\ell)), t) \\
&\quad = \left(\inf_{\ell \in [1,3]} \frac{t}{t + |\int_1^\ell (Tx(u) - Ty(u)) e^{1-u} \, du|^2}, \sup_{\ell \in [1,3]} \frac{|\int_1^\ell (Tx(u) - Ty(u)) e^{1-u} \, du|^2}{t + |\int_1^\ell (Tx(u) - Ty(u)) e^{1-u} \, du|^2} \right) \\
&\quad \geq \left(\frac{t}{t + \frac{1}{e^4} |\int_1^3 (Tx(u) - Ty(u)) \, du|^2}, \frac{\frac{1}{e^4} |\int_1^3 (Tx(u) - Ty(u)) \, du|^2}{t + \frac{1}{e^4} |\int_1^3 (Tx(u) - Ty(u)) \, du|^2} \right) \\
&\quad = \mathcal{M}_{M,N}(gx, gy, t),
\end{aligned}$$

then

$$\mathcal{M}_{M,N}\left(fx, fy, \left(\frac{t}{e^4}\right)\right) \geq_L \mathcal{M}_{M,N}(gx, gy, t).$$

Thus all conditions of Theorem 2.4 are satisfied for $\phi(t) = \frac{t}{e^4}$ and so f and g have a unique common fixed point, which is the unique solution of the integral equations

$$x(\ell) = 4 + \int_1^{\ell} \left(x(u) - u^2\right) e^{1-u}\, du$$

and

$$x(\ell) = (1-\ell)^2 e^{1-\ell} + \int_1^{\ell} \int_1^{u} \left(x(v) - v^2\right) e^{2-(u+v)}\, dv\, du.$$

Competing interests
The author declares to have no competing interests.

Author's contributions
Only the author contributed in writing this paper.

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